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METHODOLOGY FOR EVALUATION OF TASK FORCE DEFENSE CONCEPTS-CTG (CARRIER TASK GROUP) DEFENSE EFFECTIVENESS IN A MULTI-THREAT ENVIRONMENT

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Operations Research, Incorporated

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This report presents the results of the most recent research efforts (Ocotber, 1973) in a continuing investigation into the development of methodology for evaluating task force defense concepts. A surface defense model is formulated for the purpose of assessing task force defense capability against missile attack by enemy surface ships. This model is then integrated with previously developed submarine and air defense models to form one composite multi-threat model. Numerical examples are provided to illustrate potential

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model applications. Areas requiring further research and development are defined and recommendations made concerning the logical progression of future research efforts in support of the overall program objectives.

OPERATIONS RESEARCH, Inc.

SILVER SPRING, MARYLAND

METHODOLOGY FOR EVALUATION OF TASK FORCE DEFENSE CONCEPTS – CTG DEFENSE EFFECTIVENESS IN A MULTITHREAT ENVIRONMENT

By
Peter K. Luster and Wilfred Palmer

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Prepared under Contract N00014-73-C-0426
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PREFACE

This report has been prepared by Operations Research, Inc. for the Office of Naval Analysis Programs, Office of Naval Research, Department of the Navy. The results reported here represent the final report of the tasks associated with Contract No. N00014-73-C-0426.

The research reported herein forms a portion of a continuing investigation into the development of methodology for evaluation of task force defense concepts. The present study effort consists of the development of a Carrier Task Group surface defense model; the integration of this model with existing submarine and air defense models to form one composite multi-threat model; and, the evaluation of numerical examples to illustrate potential model applications.

This study has been performed under the sponsorship of the Director, Naval Analysis Programs, of the Office of Naval Research. The ONR Project Scientific Officer was CDR R. A. McCaffery, USN. Appreciation is here expressed to CDR McCaffery for his cooperation and assistance during the course of the study effort.

The following Operations Research, Inc. and consultant personnel contributed directly to the accomplishment of this study:

Dr. Samuel G. Kneale

Mr. Ben B. Levitt

Dr. Wilfred Palmer

Mr. Peter K. Luster, Consultant

EXECUTIVE SUMMARY

This study forms a portion of a continuing investigation into the development of methodology for evaluation of task force defense concepts. This work has been performed by Operations Research, Inc. (ORI) over the past several years and is under the sponsorship of the Office of Naval Research, Naval Analysis Programs (Code 431).

The long-term objective of this work is to develop a unified methodology that permits the weapons system planner to evaluate the effectiveness of the various elements of fleet defense. The interim results previously reported represent an initial formulation of such a unified methodology. This has been accomplished in the context of examining the effectiveness of fleet defense against the threat of anti-ship missiles, both air-launched and submarine-launched.

Three principal analytical models of Carrier Task Group (CTG) functional defense effectiveness have been developed. These models are extremely flexible and adaptable to a wide range of tactical, operational and technical considerations. One model measures CTG defense effectiveness against a self-contained submarine reconnaissance/attack threat. A second model measures CTG defense effectiveness against a self-contained air reconnaissance/attack threat. In this context, "self-contained" infers that the threat platforms attempt to accomplish their mission without assistance from other friendly units. The third model provides a more detailed treatment of that portion of the CTG air defense problem that deals with denial of enemy air reconnaissance efforts.

The objectives of the present study effort are twofold: 1) the development and formulation of a model to measure CTG defense effectiveness against a self-contained surface reconnaissance/attack threat; and, 2) the integration of the CTG surface, subsurface and air defense models into one overall composite model characteristic of CTG operations in a multithreat environment.

The surface defense model developed in this study effort consists essentially of three distinct analytical models, linked to one another by the defense functional parameters they share in common. These models are:

- Air barrier detection model (a modification of the previously developed air barrier engagement model), which characterizes barrier geometry, enemy penetration tactics and threat detection probabilities
- Air/surface engagement model, which characterizes both defending air and enemy surface fighting strength, and total number of enemy units surviving after engagement by available air forces
- Search model, which characterizes the search tactics of the enemy surface units, defensive capabilities of the CTG surface forces and the survival probabilities of the CTG High Value Units (HVU).

The air barrier detection model expresses results in terms of barrier detection probabilities as a function of the major parameters involved. Results of the engagement model are expressed in terms of the expected number of enemy surface units surviving the interdiction efforts of the CTG air forces. And, finally, the results of the search model are presented in terms of the probability of survival by the CTG HVU for specified periods of time.

The integrated composite CTG effectiveness model is formulated in two versions. The first of these measures the effectiveness of a CTG in combatting an attack by enemy surface and subsurface forces when these units operate independently in an attempt to destroy the HVU. The second model version measures the effectiveness of a CTG in countering a coordinated air and sea attack. In this latter instance, the enemy surface and subsurface units attempt to locate and hold contact on the HVU thereby delaying their attacks so as to have them coincide with the air attack which is assumed to occur at a predetermined time.

The combined surface and subsurface attack model is formulated in terms of Markov chain events and transition probabilities. Computational requirements include the necessity for determining the required transition

probabilities and then manipulating them in order to quantify CTG defense effectiveness. The primary thrust of the coordinated air and sea attack model is the determination of the probability that contact is held on the HVU by the enemy surface and subsurface units at the time that the air attack is to be initiated. In both versions, the principal measure of CTG defense effectiveness is the probability of HVU survival for prestated intervals of time.

The major conclusion resulting from this study is that a fundamental, unified methodology has been developed which will enable Navy systems and tactical planners to evaluate, in a multithreat environment, the effectiveness of current Naval Carrier Task Forces in successfully countering attacks by air, surface and subsurface forces; whether independently launched or coordinated.

Accordingly, based upon these and the previous methodological developments, it is recommended that:

- Research be undertaken to determine, from existing data, the range of parametric values characterizing today's systems and that these data be employed in the CTG effectiveness methodology to assess current Carrier Task Force defense capability
- The CTG effectiveness methodology be applied in such a manner as to evaluate the potential contributions to Carrier Task Force effectiveness of new and/or proposed defense systems.

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I. INTRODUCTION

BACKGROUND

This study forms a portion of a continuing investigation into the development of methodology for evaluation of task force defense concepts. The work has been performed by Operations Research, Inc. (ORI) over the past several years and is under the sponsorship of the Office of Naval Research, Naval Analysis Programs (Code 431). This previous work is documented in References 1, 2, and 3.

The long-term objective of this work is to develop a unified methodology that permits the weapons system planner to evaluate the effectiveness of the various elements of fleet defense. The interim results previously reported represent an initial formulation of such a unified methodology. This has been accomplished in the context of examining the effectiveness of fleet defense against the threat of anti-ship missiles, both air-launched and submarine-launched.

Three principal analytical models of Carrier Task Group (CTG) functional defense effectiveness have been developed. These models are extremely flexible and adaptable to a wide range of tactical, operational and technical considerations. One model measures CTG defense effectiveness against a self-contained submarine reconnaissance/attack threat. A second model measures CTG defense effectiveness against a self-contained air reconnaissance/attack threat. In this context "self-contained" infers that the threat platforms attempt to accomplish their mission without assistance from other friendly units. The third model provides a more detailed treatment of that portion of the CTG air defense problem that deals with denial of enemy air reconnaissance efforts.

The Submarine Defense Model

The purpose of the analysis of CTG defense effectiveness against the self-contained submarine reconnaissance/attack threat is to measure the relative significance of new systems and operational tactics for degrading the ability of the submarine force to target high-value CTG elements. If the CTG elements are only known to be located somewhere in a large deployment area, and CTG radiations are carefully controlled, then the submarine force must infiltrate the operations area, and conduct a more-or-less random search (depending on available intelligence) for these elements, which may be dispersed throughout the area. Even if a ship formation (or simulated formulation) is detected, it is necessary for a submarine to spend time in its vicinity classifying its elements in order to develop a fire control solution against any high-value units (HVU) in the formation. Thus, the presence of credible low-value units (LVU) in the operations area increases the chance that a given submarine will waste its fire on an LVU, and expose itself to post-attack ASW counter-action. Moreover, throughout the search and the target classification period, the submarine is subject to counter-detection and destruction. Thus, dispersed CTG dispositions, perhaps enhanced by non-ship decoy targets, force the enemy to detect and correctly classify LVUs. This tactic simultaneously

- Reduces the effective threat force search rate for the HVU
- Increases the risk of a searching submarine's destruction at an LVU before encountering the HVU.

Moreover, submarine forces that are small, relative to the number of potential credible targets, must pay for numerical weakness with an extended classification of CTG targets in their lethal vicinity in order not to waste their limited firepower. This increases the effectiveness of the HVU's local ASW defenses. Finally, the search, attack, and counter-attack process continues in time, and the likelihood of successful HVU survival must be related to the necessary duration of the CTG primary mission in the operations area.

There are a number of issues which suggest themselves. How do the numbers of LVU, their local ASW capability, and their "credibility" (the time required to classify them, and the probability they will be mis-classified) effect the probability of HVU survival through time? What is the significance of the detectability and ease of classification of the HVU and the strength of its local ASW defense? For a given CTG target composition, how does HVU target survival vary in time with the number of enemy submarines and the search rates for low- and high-value targets? The functional model of CTG defense effectiveness presented next provides an integrated means for answering such questions.

Suppose the BLUE force operates Y LVUs and one HVU in an Area A. The LVUs are assumed to be equally effective in their submarine kill potential, in their credibility in terms of the mean time taken by a submarine to reach a classification decision, and in the probability that this decision is incorrect and the submarine will expend itself in attack. Assume that RED does not wish to attack LVUs. All CTG targets are assumed randomly and uniformly distributed in position and velocity throughout the operations area. The submarine rate of encounter with the HVU may differ from its rate of encounter with a single LVU, however. Assume that X RED submarines randomly and independently search the deployment area for the HVU formation[1/]. There is no communication or coordination between submarines. As a criterion for BLUE mission success, let $V(t|X,Y)$ denote the probability that the HVU survives t days of search.

Any LVU is characterized by three functional parameters that may be jointly calculated from detailed tactical engagement models, or determined experimentally. These are:

σ: Probability that a submarine, on encountering an LVU, will not be destroyed by the local ASW defense before the target is classified. (This probability is unity in the case of pure-delay type LVUs that RED never mis-classifies, and in the case of COLD-war scenarios).

δ: Probability that a submarine that survives an encounter with an LVU will correctly classify the target. With probability $1 - \delta$, the submarine mistakenly attacks and is removed as an HVU threat.

h: Mean time (days) needed by a submarine that survives an encounter with an LVU, to make a target classification decision (which will be correct with probability δ).

The HVU formation is also described by the same parameters σ_o, δ_o, h_o, which may differ from σ, δ, or h. Then, with these data and assumptions,

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[1/] This may not be the most effective RED search strategy. RED, for example, might want to divide the deployment area into X parts, one per submarine, search all, or most, of each part and continually cycle submarines through the parts. Preliminary studies show that this particular search strategy is not optimal for the criterion of mission success adopted here, but this is a topic area for further study. Uniform random search is likely to be close to optimum regardless of the criterion adopted.

the probability that the HVU will survive t days of search is given by

$$V(t|X,Y) = \left[\frac{YK+K_o}{YK+E_o} + \frac{\delta_o\sigma_o}{YK+E_o}\exp\left(-\frac{t(YK+E_o)}{T_o+hY(1-K)+h_o(1-E_o)}\right)\right]^X$$

where

- $K = 1 - \delta\sigma$, the probability an LVU will either kill or exhaust a submarine on encounter
- $E_o = 1 - (1-\delta_o)\sigma_o$, the probability a submarine will either kill, or be killed by, the HVU on encounter and not mis-classify it and pass by
- $K_o = 1 - \sigma_o$, the probability that the HVU will destroy a submarine on encounter before it makes a classification decision
- T_o = Time required to search Area A at the submarine's effective sweep rate against a quiet HVU.

The formula essentially describes the CTG functional capability to defend against the submarine launch platforms themselves. Indeed, this formula assumes that if a submarine encounters the HVU, survives its local ASW defense, and correctly classifies it, then the subsequent missile attack will destroy the HVU. The HVU terminal missile defenses, however, might defeat such an attack.

The HVU functional capability for terminal missile defense can be included in the formula explicitly as follows. Let μ_o denote the probability that the HVU terminal defense can successfully intercept, deflect, or absorb the launched cruise missiles. The functional missile defense parameter μ_o depends upon a number of technical parameters that characterize the active and passive terminal missile defense systems, and cruise missile capabilities. Then, the formula for HVU survival through time becomes

$$V(t|X,Y) = \left[\frac{YK+K_o+\delta_o\sigma_o\mu_o}{YK+E_o} + \frac{\delta_o\sigma_o(1-\mu_o)}{YK+E_o}\exp\left(-\frac{t(YK+E_o)}{T_o+hY(1-K)+h_o(1-E_o)}\right)\right]^X$$

Consequently, the improvement in HVU survivability due to the functional capability for terminal missile defense can be treated explicitly.

<u>Properties of the HVU Survival Function—LVU of Pure-Delay Type</u>. The formula for the probability that the HVU will survive t days of search by X submarines is the sum of a steady-state probability of survival and a time-dependent probability (that exponentially falls to zero with time), raised to the power X. The total effective classification holding time developed by the LVU force is given by $H = hY(1-K) + h_o(1-E_o)$. It follows from the formula that in order to have any significant effect on HVU survivability, H must be large relative to T_o.

Thus, the formula implies that the reduction in submarine effective search rate imposed by LVU classification holding times has a small effect on improving the probability of HVU survival through time, unless the characteristic submarine tour time T_o is small. This latter case is of little interest in the context of this discussion.

Thus, the formula can be well-approximated by

$$V(t\,|\,X,Y) = \left[\frac{YK+K_o}{YK+E_o} + \frac{\delta_o\sigma_o}{YK+E_o}\exp\left(-\frac{t(YK+E_o)}{T_o}\right)\right]^X$$

which slightly underestimates the probability of HVU survival through time.

The Air Defense Model

This section continues the development of analytical models of CTG functional defense capabilities with a summary description of a quantitative analysis of CTG defense effectiveness against the bomber reconnaissance/attack threat in a HOT-war scenario.

The purpose of this analysis of CTG defense effectiveness against the bomber threat is to examine the significance of a target intelligence defense against this threat, and to measure the relative contributions to overall CTG defense effectiveness of the three CTG functional capabilities for intelligence, platform and terminal missile defense.

The general theme of the preceding discussion of the submarine threat repeats itself here. If the locations of CTG high-value units are not sufficiently well known to the enemy, then he is forced to use aircraft in a reconnaissance role in order to locate and classify the HVU. Thus, threat aircraft are committed to search and attack on terms that are more favorable to the CTG forward air defenses. Assuming that the CTG maintains a long-range air surveillance and tracking capability, interceptors can be selectively committed against enemy bomber attacks with the option of not engaging misdirected raid groups. This mis-direction of search/attack aircraft, due to the lack of targeting intelligence, reduces the expected number of bombers that penetrate to the HVU, the number of air-to-surface missiles with which the HVU formation terminal defense must contend, and increases the ratio of interceptors to bombers in the defense of the HVU formations.

In certain essential respects, the submarine and air threats are polar opposites. The covert nature of the submarine threat means that the CTG must provide a general level of ASW submarine kill capability throughout the deployment area, without knowing the locations and motions of searching submarines. With some probability, a submarine will encounter these defenses and be destroyed, or will eventually locate, classify, and attack the HVU. Each submarine faces essentially the same risk of destruction, so that, on the average, a constant fraction of the enemy force will be destroyed, whatever its size.

Thus, covert submarine attack forces are characterized as "percentage vulnerable." Low submarine search rates and their poor ability to communicate with each other in the search mode imply that attacks on the HVU occur one-at-a-time, and not in a massed synchronized strike from many cooperating submarines.

In contrast, air reconnaissance and attack are overt, in the sense of being observable at long ranges. Hence, defensive interceptors and later, close-in SAM fire-power, are spread across known numbers of bombers. In this case, the ratio of defensive missiles (AAM and SAMs) fired per attacking bomber is the main determinant of bomber survival. The smaller this ratio, the larger the probability that an individual bomber will survive. Thus, overt bomber attack forces are characterized as "numerically vulnerable." Consequently bomber forces are motivated to attack synchronously and in mass in order to saturate finite air and missile defense capabilities and minimize the interceptor missile/bomber ratios. Their reliable air-to-air communications facilitates such attack coordination.

Despite these striking contrasts in the nature of the vulnerabilities and operating modes of the submarine and bomber threats, the purpose of a CTG target intelligence defense capability remains that of forcing the enemy to spread his attack across many credible targets and suffer attrition at places of low value. In the absence of a functional capability to deny the enemy targeting intelligence on the HVU formation, this formation would have to protect itself against a synchronized attack from every available enemy platform.

A CTG targeting intelligence defense capability is achieved by

- Creating several credible HVU formations in the CTG operating area
- Defending these formations against localization by enemy search units
- Preventing their correct classification on being detected by search units.

The object of intelligence defense is to force the enemy to expend aircraft and resources for reconnaissance on terms favorable to the CTG forces, and to divide limited attack resources across a number of unclassified LVUs, this division of enemy resources reduces the effective weight of attack against the HVU. Regardless of the means by which credible target formations are created (physically dispersing units, decoys, ECM), the enemy is confronted with two general types of reconnaissance situations: strategic reconnaissance and tactical reconnaissance.

The Air Reconnaissance Defense Model

The purpose of a defense against enemy air reconnaissance of the CTG objective area is to force the enemy to expend resources to obtain the

timely target location and classification data necessary for the most effective use of bomber (or submarine) attack forces. With imperfect prior target intelligence attack forces themselves, must first search for suitable CTG targets and thereby expose themselves to counter-action while operating in a more vulnerable, and less lethal, reconnaissance mode. For example, if a force of attack bombers must first penetrate a large defended area and then use radar to search for targets of opportunity, their concentration against a dispersed target set is disrupted; their location is continuously revealed; there is little time to classify targets once detected; and sub-optimum approaches to fire-control system lock-on and weapon release lines may be required. Consequently, prior air reconnaissance of the objective area to locate and, if possible, classify targets and designate these to a follow-on bomber force (or in-place submarine force) is indicated.

For example, assume that the CTG (BLUE) has dispersed several real, or decoy, target formations in an objective area some hundreds of miles in diameter. The enemy (RED) must

- Fly a force of reconnaissance (recce) aircraft perhaps accompanied by fighter escort, to the objective area
- Penetrate an active air defense barrier line with sufficient aircraft to locate and, if possible, classify BLUE targets
- Deliver this target intelligence data back to home base, or to a bomber force trailing the reconnaissance attack by a distance short enough to prevent any significant degradation of the intelligence before their arrival or to a ship/sub attack force.

Thus, BLUE must balance his commitment to a strong airborne reconnaissance defense against the need for a reserve of AEW and fighter aircraft to meet potential follow-on bomber attacks.

There are essentially three major components to this air defense problem, the characteristics of which must be represented in the analytic functional defense effectiveness model. These are

- <u>BLUE CTG RECCE DEFENSE DISPOSITION</u>
  BLUE's selection and occupation of an area of strategic uncertainty large enough to present RED with an air reconnaissance problem; the specification of an air defense barrier relative to this objective area; and the determination of a CTG disposition, EMCON, and ECM posture.

- RED RECCE THREAT
  RED's choice of the number of reconnaissance and escort fighter aircraft to commit, and a barrier penetration tactic.

- BLUE INNER DEFENSES
  BLUE's reserve defensive response to barrier detections and penetrations, and the initiation of RED radar surveillance.

These components of the air defense problem, as well as those associated with the submarine reconnaissance and attack problem have been incorporated in the models resulting from the prior research efforts. As previously stated, the remaining tasks are: to formulate a model describing the surface engagement; and, to integrate all of the models with a single multi-threat environment.

STUDY OBJECTIVE

As indicated above, the previous study efforts have produced models that are independently capable of handling either the air or the submarine threat. The logical extension to this work is to develop a methodology that will permit simultaneous handling of a combined threat. This would be representative of a more realistic operational scenario. Accordingly, the principal objective of the current study effort is to develop a methodology for evaluation of task force defense concepts in a multi-threat environment.

STUDY APPROACH

The most important aspect of CTG survivability that has not yet been modeled is its defense against missile attack from enemy surface ships. The basic study approach, then, is to develop a CTG surface defense model compatible with the existing submarine and air defense models, and to aggregate these individual models into one composite multi-threat model.

The study approach consists of the following five major tasks:

- Identify the engagement rules, tactical considerations and operational parameters involved in the surface defense of a Carrier Task Group

- Formulate a CTG surface defense model that characterizes the considerations identified above

- Examine the interactions that occur between the submarine, air and surface defense models, and review the four models for compatibility

- Aggregate the individual models into one overall composite model characteristic of the CTG operating in a multi-threat environment
- Exercise the multi-threat model.

REPORT ORGANIZATION

Section II of this report is confined only to documentation of the CTG surface defense model, the first two tasks described previously. In the development of such a model, as with the submarine and air defense models, it is postulated that the CTG is being placed under attack only by a single type of threat. Thus, at this stage of the study effort a number of important considerations are intentionally by-passed. These are primarily concerned with the decisions that must be made by the CTG commander in allocation of his defensive resources in response to an enemy surface attack. They include such questions as the number of attack aircraft (VA) that should be employed to counter a detected enemy surface raid, and the number of repeated attacks that should be made; CTG action to be taken when it is known that enemy surface elements have penetrated the AEW screen; and the use of available forces to conduct open ocean search for enemy surface ships within the CTG operating areas.

It is recognized that such decisions must be based on a full assessment of the threat to which the CTG will be exposed from all sources. Thus, the surface defense model, taken alone, might appear to be operationally unrealistic since it is unlikely that the CTG would ever be in a situation where the only threat consisted of surface forces. Consideration of these multi-threat factors is made in Sections III and IV of this report.

Supporting and background mathematical developments are presented in the appendices.

An additional constraint should be observed in regard to the numerical examples presented in this report. These examples are meant only to demonstrate the methodology being developed, and are not based on operationally significant data. This has been done to keep the report at an unclassified level.

II. STRUCTURE OF THE SURFACE DEFENSE MODEL

OVERVIEW OF THE SURFACE DEFENSE MODEL

The Carrier Task Group (CTG) tactical situation structured for this study is identical to that used previously in the development of the submarine and air defense models. Basically this consists of the CTG performing the power projection ashore mission while operating in a constrained area within the proximity of its assigned land targets. The enemy forces know the location of the CTG only with some uncertainty. Therefore enemy attack forces sent against the CTG must accomplish some degree of reconnaissance in order to establish the position of the High Value Unit (HVU) within the strategic area of uncertainty.

The CTG allocates its defensive resources in two different ways. A portion of the available CTG forces is assigned to screening functions which are in direct defense of an attack on the HVU. Other CTG defensive resources are allocated to support functions outside of the HVU screen but inside the strategic uncertainty area. These external CTG forces also serve to attrite those enemy forces attempting to pinpoint the location of the HVU.

For the purpose of the surface defense model it is assumed that the enemy forces must also enter the strategic uncertainty area and conduct a search to determine the precise location of the HVU, or at least to determine its position with sufficient accuracy to launch a missile attack against it. Thus enemy surface forces entering the strategic uncertainty area will be exposed to detection by the CTG AEW aircraft in an established air barrier. Enemy surface ships detected by the air barrier will be subjected to attack by VA aircraft launched from the CV and vectored to their position by the AEW barrier aircraft. Those surface ships that either are not detected by the air

barrier or survive attack from VA aircraft enter the strategic uncertainty area and initiate their search for the HVU. During the time that the enemy surface ships are within the strategic uncertainty area, they are still exposed to possible detection when within range of the AEW barrier or when they come within contact range of external CTG forces operating in the area. Finally, the surface ships are exposed to detection and attack by CTG ASW forces operating as part of the HVU screen.

Development of the surface defense model is approached here as if the CTG were defending itself against surface attack only. In a later section of this report consideration will be given to simultaneous attack by enemy surface, air, and submarine forces.

For the present model it is assumed that the threat to the CTG is from shipborne guided anti-ship missiles. There are four elements to the CTG defensive capabilities against this threat, as follows:

1. Deny the enemy targeting information by random motion of the HVU within the strategic uncertainty area, and employ external search forces to confuse and decoy the enemy

2. Detect the enemy's ships as they come within range of the CTG air barrier and destroy them with VA aircraft

3. Detect the enemy's near approach (penetration through the HVU screen) and attack him with VA aircraft

4. Destroy, deflect, or absorb the enemy's surface launched missiles that have escaped the CTG defensive forces.

These four items have been treated parametrically in earlier work in either the CTG submarine defense model or in the CTG air defense model. Relevant portions of these earlier models have been reviewed and adapted for the surface defense model.

The early detection/penetration situation is depicted in Figure 1, and a functional block diagram for the surface defense model in its entirety is given in Figure 2. As the enemy surface ships approach the CTG AEW air barrier they may or may not be detected. If detected they are engaged by CTG attack (VA) aircraft vectored to their position. The enemy surface ships may or may not survive this engagement or they may be subjected to repeated attacks by VA aircraft. If undetected they advance within the strategic area

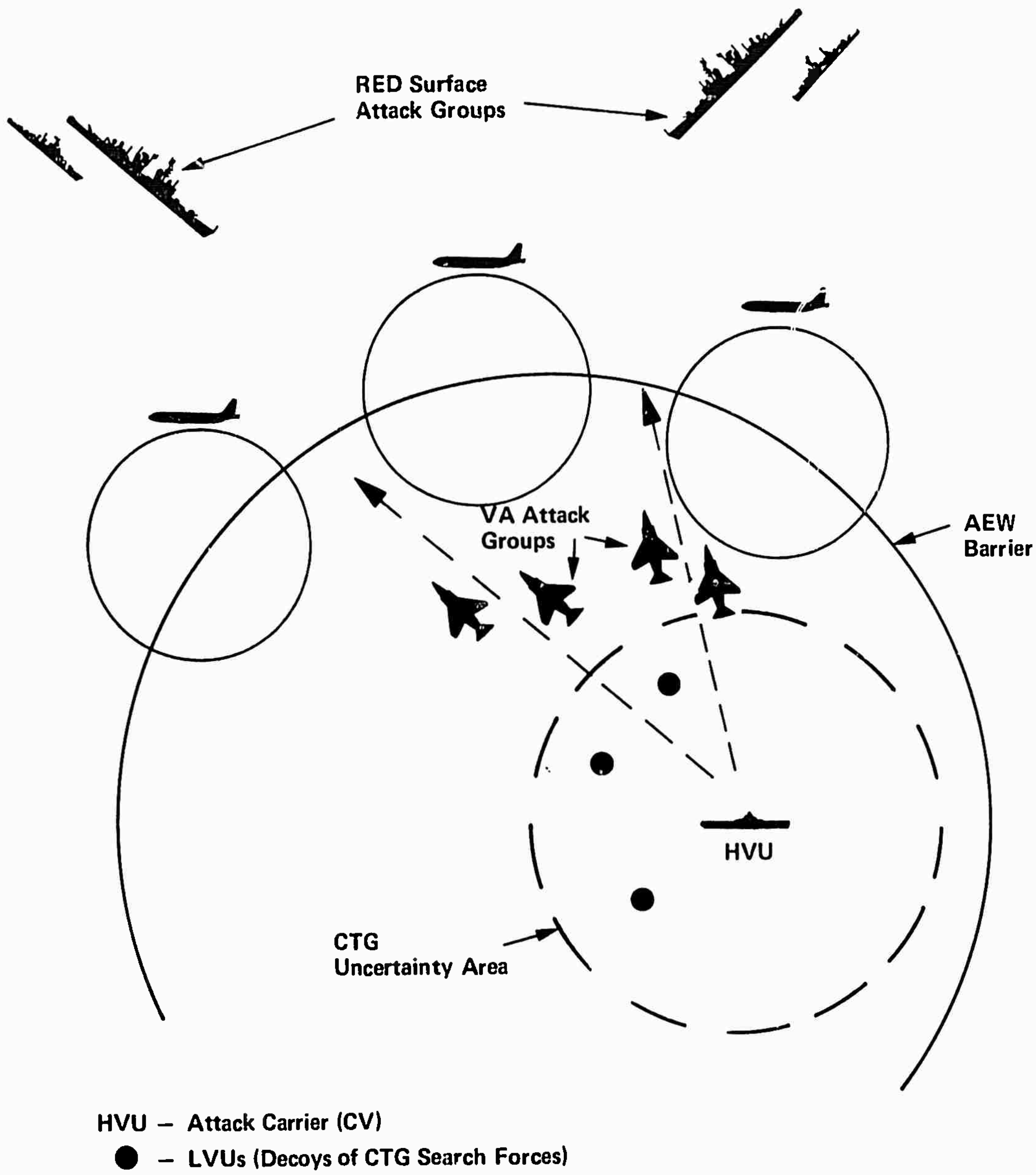

FIGURE 1. SURFACE DEFENSE MODEL OVERVIEW

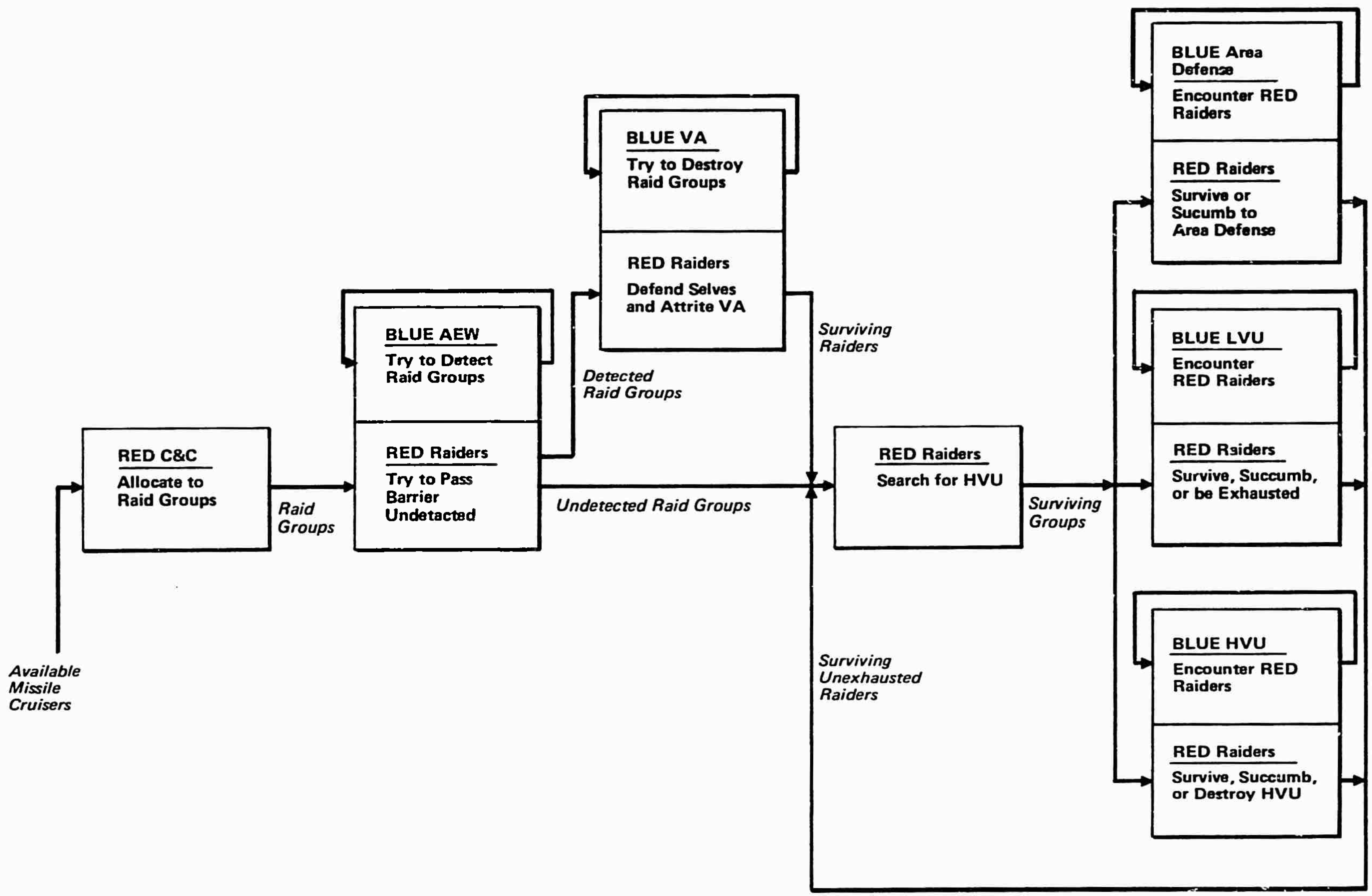

FIGURE 2. SURFACE DEFENSE MODEL FUNCTIONAL BLOCK DIAGRAM

of uncertainty and commence a search for the HVU. The detection of enemy missile cruisers attempting to penetrate the AEW barrier is described mathematically in the subsection concerning the Air Barrier Detection Model. The subsequent engagement of detected missile cruisers is described mathematically in the subsection concerning the Air/Surface Engagement Model; two versions of this model are given, the second, more detailed model being available for independent use or, if desired, for performing sensitivity analyses on the first model.

Enemy surface forces that enter the CTG uncertainty area undetected, or those that have survived attack by VA aircraft, are faced with the problem of seeking out the HVU. During this time they are subject to detection and attack and also to decoying actions by the CTG defensive forces allocated to this function. The searching surface ships employ radar (and ESM) as their primary detecting sensors, and the CTG forces defending the HVU employ radar and ESM to detect the enemy surface forces. This engagement is characterized by the Search Model. Development of this submodel is described in a later subsection.

In summary, the basic approach of the surface defense model is to characterize the penetration into the strategic uncertainty area by the Air Barrier Detection Model and the Air/Surface Engagement Model. Once the surface forces have entered the area of uncertainty area where the CTG is operating, the Search Model is used to determine the probability that the CTG is able to avoid attack by the enemy surface forces.

An exposition of the surface defense model is given in the following subsection, in the context of a numerical example. It is hoped that, thus presented, the model will be more readily grasped by the intuition than with the more detailed and formal presentation alone.

NUMERICAL EXAMPLE

The purpose of the present numerical example is to help explain the Surface Defense Model and to illustrate the sort of analysis that can be carried out using it. The numerical values used in this example are purely hypothetical, and so no conclusions of any operational significance can be drawn from model results based on these numerical values.

The nature and disposition of the BLUE forces is, within the Model a <u>fait accompli</u>. That is, the model results may be used to evaluate any given BLUE strategy regarding the nature and disposition of his forces; but the effects of such strategy enter into the model as inputs, and possible BLUE alternatives are not discussed within the model. An apparent exception is BLUE's decision about how to deal with raid groups detected at the AEW barrier. It turns out

that (owing to the special assumptions made) this decision does not affect the results of the present model, although it will be of some importance in the later model combining air, surface, and submarine models.

RED, on the other hand, must decide how to split up his X available missile cruisers into raid groups, where to have them attempt to penetrate the AEW barrier, and how to have them search for the HVU. Certain other RED decisions, such as the policy to be used in attacking a suspected HVU that could in fact be an LVU, will be reflected in the values of the relevant parameters and may therefore, like BLUE's strategic decisions, be regarded as taking place outside the scope of the model. The RED decisions made within the scope of the model are interrelated in a manner that will become clear later.

The available RED missile cruisers are assumed to be all alike, so far as their capabilities as reflected in the present model are concerned. Therefore to calculate the number of ways of splitting them up into raid groups one does not take account of which cruisers go into which raid groups, but only of how many raid groups there might be consisting of just one cruiser, how many consisting of just two cruisers, etc. For instance, for $X = 4$ there are just five essentially different raid group formations. Using the notation of partitions, these five possibilities may be enumerated as follows:

Notation	Raid Group Description
4	A single raid group, consisting of all four cruisers
31	One group with three cruisers and one group with one cruiser
2^2	Two groups of two cruisers each
21^2	One group of two cruisers and two groups of one cruiser each
1^4	Four groups of one cruiser each.

Given X, the number of essentially different raid group formations that can be formed is known mathematically as the number of partitions of X, taken without restriction. A graph of the number of partitions of X as a function of X is given in Figure 3; this graph shows that the number of partitions of X is modest for small values of X, but that it grows rapidly thereafter. For instance, there are only 42 partitions of 10, but there are 10,143 partitions of 33.

According to the Air Barrier Detection Model, the probability $\epsilon(X_i)$ of BLUE detecting a raid group of X_i cruisers at the AEW barrier depends on X_i, but does not depend on where in the barrier the attempted penetration is made. Moreover, the probabilities of detecting the various raid

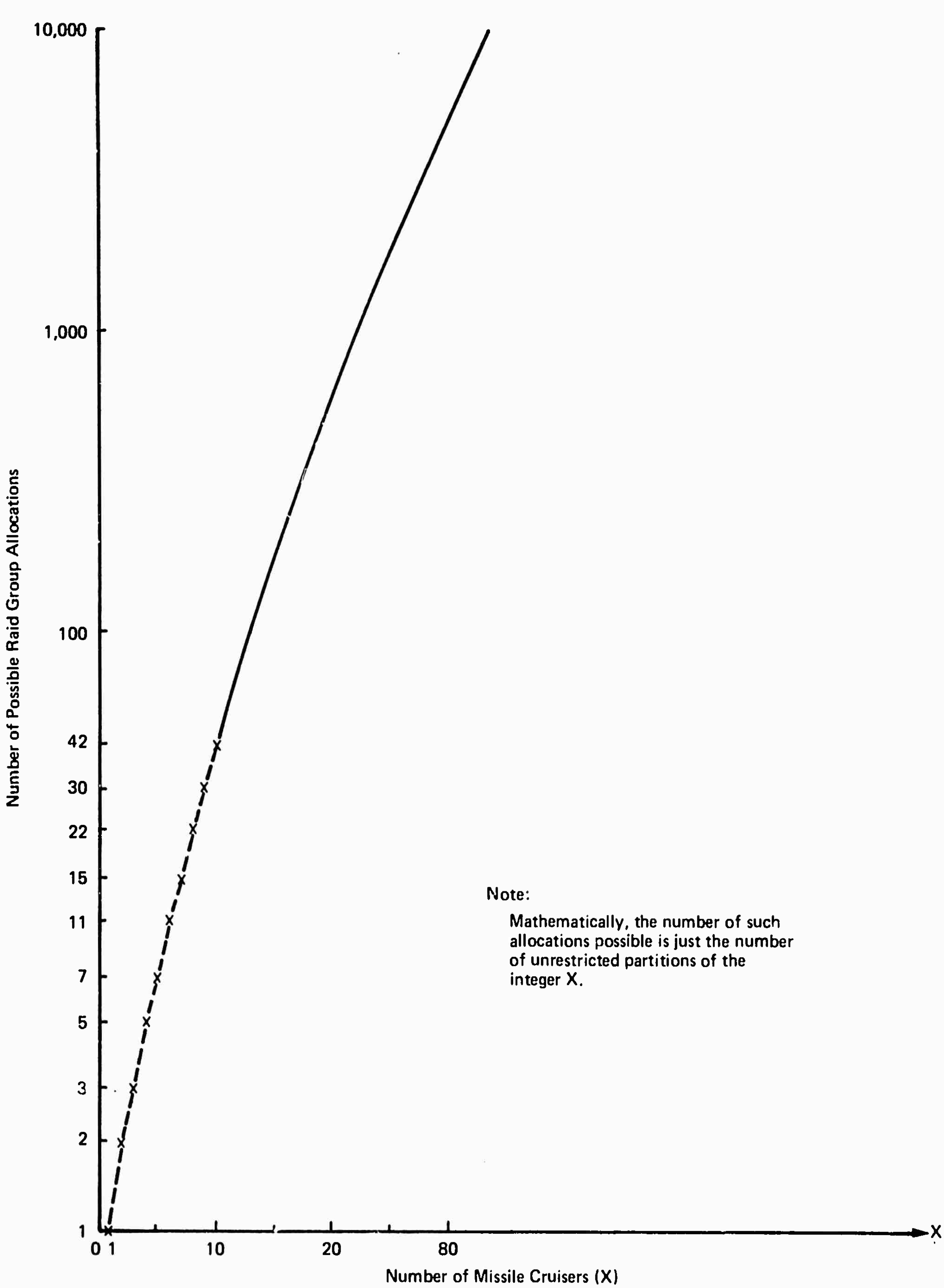

FIGURE 3. NUMBER OF POSSIBLE ALLOCATIONS OF RED MISSILE CRUISERS INTO RAID GROUPS

groups are assumed to be mutually independent. In the four-cruiser example the probabilities of detecting each possible number of cruisers may therefore be tabulated, as a function of the chosen formation into raid groups, as shown in Table 1. Supposing for simplicity that $\epsilon(X_i) = .6$ for $X_i \in \{1,2,3,4\}$. This then leads to the numerical results tabulated in Table 2.

Detected raid groups suffer attrition from the CAP in accordance with the provisions of the Air/Surface Engagement Model. This model also determines how much attrition the VA aircraft suffer in attacking the raid groups. Which defected raid groups BLUE attacks, when, and with how much force are decisions that will be discussed in connection with the combined model.

Numbers of RED and BLUE survivors in any given VA attack on a raid group are given by Equations (7) and (8) (see Air/Surface Engagement Model Subsection) as functions of the numbers of participants. For illustrative purposes only, suppose that a four-aircraft VA contingent attacks a raid group consisting of four missile cruisers; that the surviving VA aircraft attack the surviving missile cruisers; and so on, until one or both forces are annihiliated. (That at least one of the forces will, in the limit, be annihilated is shown in Appendix A.) Suppose the parameters α and β are defined implicitly by exp $(-\alpha) = .5$ and exp $(-\beta) = .9$. Approximate numbers of survivors of the first few attacks are then as follows:

	RED	BLUE
Initial force levels	$X(0) = 4$	$Y(0) = 4$
Survivors of first attack	$X(1) = 2$	$Y(1) = 3.6$
Survivors of second attack	$X(2) = .57$	$Y(2) = 3.40$
Survivors of third attack	$X(3) = .01$	$Y(3) = 3.34$.

As the number of attacks is increased indefinitely, the number of RED survivors approaches zero. The number of BLUE survivors approaches a limit that can be estimated to within any desired degree of accuracy by the methods of Appendix A.

To illustrate the Search Model, suppose that exactly $X = 1$ missile cruiser escapes detection at the AEW barrier. Moreover, suppose that all barrier-detected cruisers pose no threat to the HVU, either because they will be destroyed by the VA, or for some other reason. Suppose that there is just one type of LVU and that there is no general area force. Suppose the other parameter values to be as follows:

TABLE 1

PROBABILITY THAT THE UNDETECTED RAID GROUPS WILL MATCH A SELECTED COMPOSITION

R	Description of Raid Group Composition									
	4		31		2^2		21^2		1^4	
	A	B	A	B	A	B	A	B	A	B
4	4	$1-\epsilon(4)$	31	$(1-\epsilon(3))(1-\epsilon(1))$	2^2	$(1-\epsilon(2))^2$	21^2	$(1-\epsilon(2)(1-\epsilon(1))^2$	1^4	$(1-\epsilon(1))^4$
3			3	$(1-\epsilon(3))\,\epsilon(1)$			21	$2(1-\epsilon(2)\epsilon(1)(1-\epsilon(1))$	1^3	$4\epsilon(1)(1-\epsilon(1))^3$
2					2	$2\epsilon(2)(1-\epsilon(2))$	2 1^2	$(1-\epsilon(2))\epsilon^2(1)$ * $\epsilon(2)(1-\epsilon,1))^2$	1^2	$6\epsilon^2(1)(1-\epsilon(1))^2$
1			1	$\epsilon(3)(1-\epsilon(1))$			1	$2\epsilon(2)\epsilon(1)(1-\epsilon(1))$	1	$4\epsilon^3(1)(1-\epsilon(1))$
0	ϕ **	$\epsilon(4)$	ϕ	$\epsilon(3)(\epsilon(1))$	ϕ	$\epsilon^2(2)$	ϕ	$\epsilon(2)\epsilon^2(1)$	ϕ	$\epsilon^4(1)$

R — Total number of missile cruisers in the undetected raid groups

A — Selected composition of undetected raid groups

B — Probability that the composition occurs

ϵ — Probability that the AEW barrier detects a raid group

Notes

* When the initial raid group formation is 21^2, the probability that the total number of missile cruisers will be 2 is $(1-\epsilon(2))\,\epsilon^2(1) + \epsilon(2)\,(1-\epsilon(1))^2$, the sum of the probabilities that 2, and that 1^2, will describe the undetected groups.

** The mathematical notation "ϕ" for the null set is used to describe the situation where none of the raid groups remains undetected.

TABLE 2

PROBABILITY THAT THE TOTAL NUMBER OF UNDETECTED RED MISSILE CRUISERS IS NOT LESS THAN A SELECTED NUMBER

R	Description of Raid Group Composition														
	4			31			2^2			21^2			1^4		
	A	B	C	A	B	C	A	B	C	A	B	C	A	B	C
4	4	.4	.4	31	.16	.16	2^2	.16	.16	21^2	.064	.064	1^4	.0256	.0256
3			.4	3	.24	.40			.16	21	.192	.256	1^3	.1536	.1792
2			.4			.40	2	.48	.64	2 1^2	.144 .096	.496	1[illegible]	.3456	.5248
1				1	.24	.64			.64	1	.288	.784	1	.3456	.8704
0	ϕ	.6	1.	ϕ	.36	1.	ϕ	.36	1.	ϕ	.216	1.	ϕ	.1296	1.

R — Total number of missile cruisers in the undetected raid groups

A — Selected composition of undetected raid groups

B — Probability that this composition occurs for $\epsilon(X_j) = 0.6$

C — Probability that the total number of undetected cruisers is at least R

$\lambda_1 = .004$ $\qquad\qquad$ $\lambda_o = .002$

$\sigma_1 = .91$ $\qquad\qquad$ $\sigma_o = .87$

$\delta_1 = .94$ $\qquad\qquad$ $\delta_o = .13$

$h_1 = .25$ $\qquad\qquad$ $h_o = .25$

$U = 200$ nautical miles per day.

These assumed numerical values yield $K = .125$ and $E_o = .25$.

Using the Search Model, the probability that the HVU will survive any given number of days may be computed. Likewise the probability that the cruiser will be destroyed within t days and the probability that the cruiser will be exhausted with 5 days may be computed as a function of t.

The values of these probabilities during the first day are sketched in Figure 4. The bottom curve in Figure 4 shows the probability that the cruiser has been destroyed as a function of time t. The second curve, which coincides with the bottom curve for $0 \le t \le \frac{1}{4}$, shows the probability that the cruiser has been destroyed or exhausted as a function of t. Accordingly, for any given t the difference between the ordinates at t to the bottom and second curves, respectively, gives the probability that the cruiser has been exhausted at time t. The third curve gives the probability that the cruiser has been destroyed or exhausted, or continues the search—i.e., the probability that it is still operational and has not yet attacked the HVU—as a function of t.

The regions between the curves are labelled to show what probabilities are represented by the differences between the ordinates bounding the regions. The region lying between the third curve and the (dashed) line representing probability identically equal to 1 is labelled "HVU destroyed", since the probability that the HVU has been destroyed 1 minus the ordinate to the third curve. A similar remark applies to the region labelled "cruiser destroyed".

Note that the probability that the cruiser has been exhausted becomes positive only after $t = \frac{1}{4}$. This is because of the assumption that the cruiser will become exhausted during an encounter only $h_1 = \frac{1}{4}$ day after commencement of that encounter, if at all. A similar remark applies to the probability that the HVU has been destroyed.

At the end of the first day the probability of each of the enumerated events is as follows:

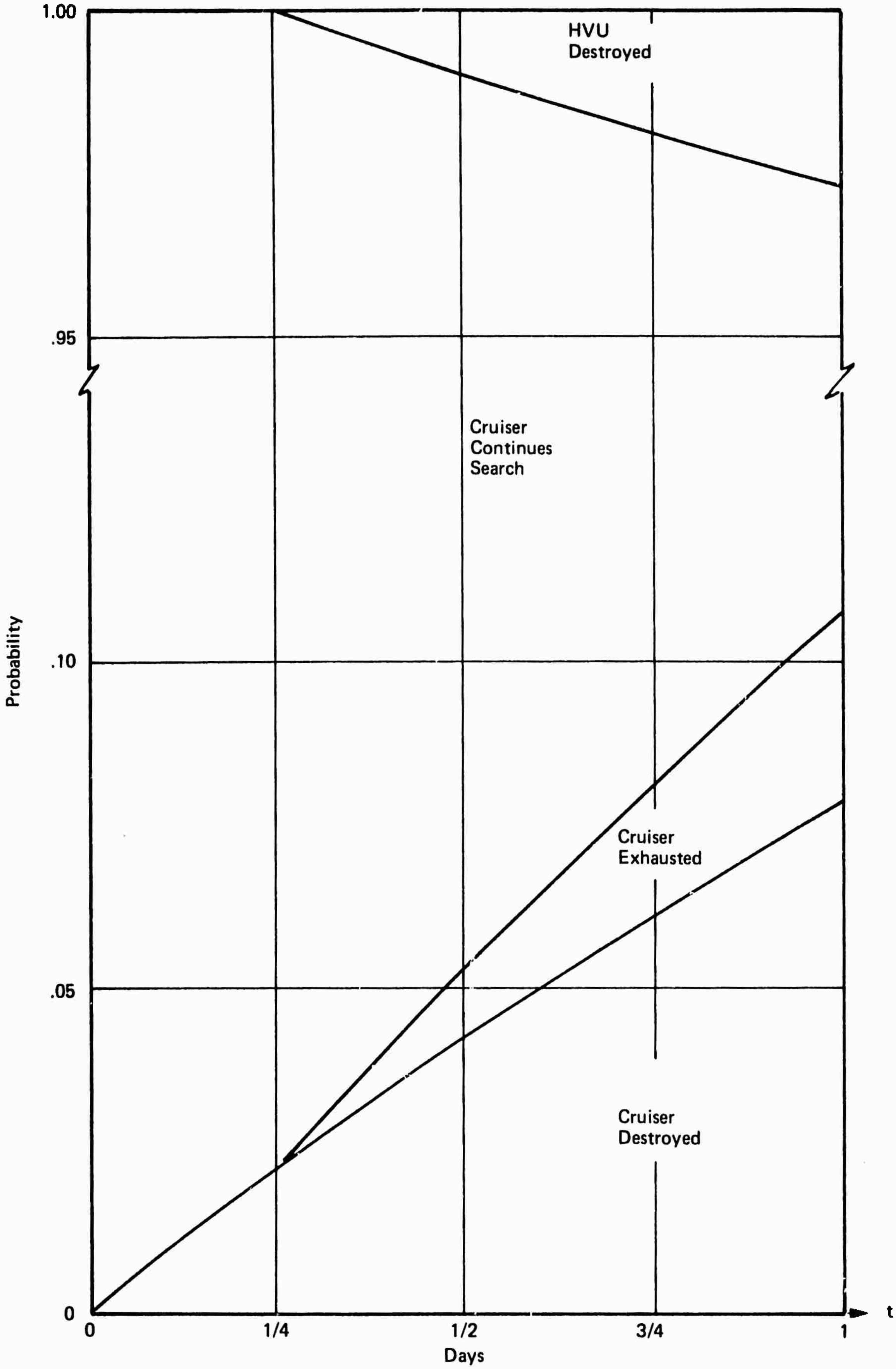

FIGURE 4. PROBABILITIES OF THE SEARCH/DESTROY EVENTS

Event	Probability at End of First Day
The cruiser has been destroyed.	.08
The cruiser has been exhausted.	.03
The cruiser is continuing its search.	.87
The cruiser has destroyed the HVU.	.03
Total (including effect of round-off errors)	1.01

As $t_0 \to \infty$ the probabilities of these events approach the following limits.

Event	Limiting Probability as $t \to \infty$
The cruiser has been destroyed.	.55
The cruiser has been exhausted.	.24
The cruiser is continuing its search.	0
The cruiser has destroyed the HVU.	.23
Total	1.00

If the number X of undetected RED cruisers at the barrier exceeds 1, the probabilities of the foregoing events may be determined from their probabilities in the X = 1 case, (see subsection on optimization). Figure 5 shows the HVU survival probability as a function of time of undetected RED cruisers at the AEW barrier, up to X = 8.

Figure 6 shows the probability that all X missile cruisers have been destroyed or exhausted as a function of t and X. Curves for $X > 3$ are too close to the t axis to be drawn on the chosen scale.

Figure 7 shows the limiting probabilities of HVU survival as $t \to \infty$ for X = 1(1)8. Here the curves which have already been shown in Figure 5 are sketched again, on a reduced scale, and then connected by dashed straight lines to the appropriate limiting values shown on the right of the figure.

Under the assumptions discussed in the subsections describing the various submodels, the initial formation of X RED missile cruisers into raid groups which maximizes the probability of destroying the HVU within <u>any</u> prescribed length t of time is 1^X (using the foregoing partition notation). This allocation is independent of t, and so an allocation that is optimal for some specific value of t will likewise be optimal for each and every other possible value of t. It is to be emphasized that this conclusion is for the RED surface threat acting alone, with no consideration given to the RED submarine and

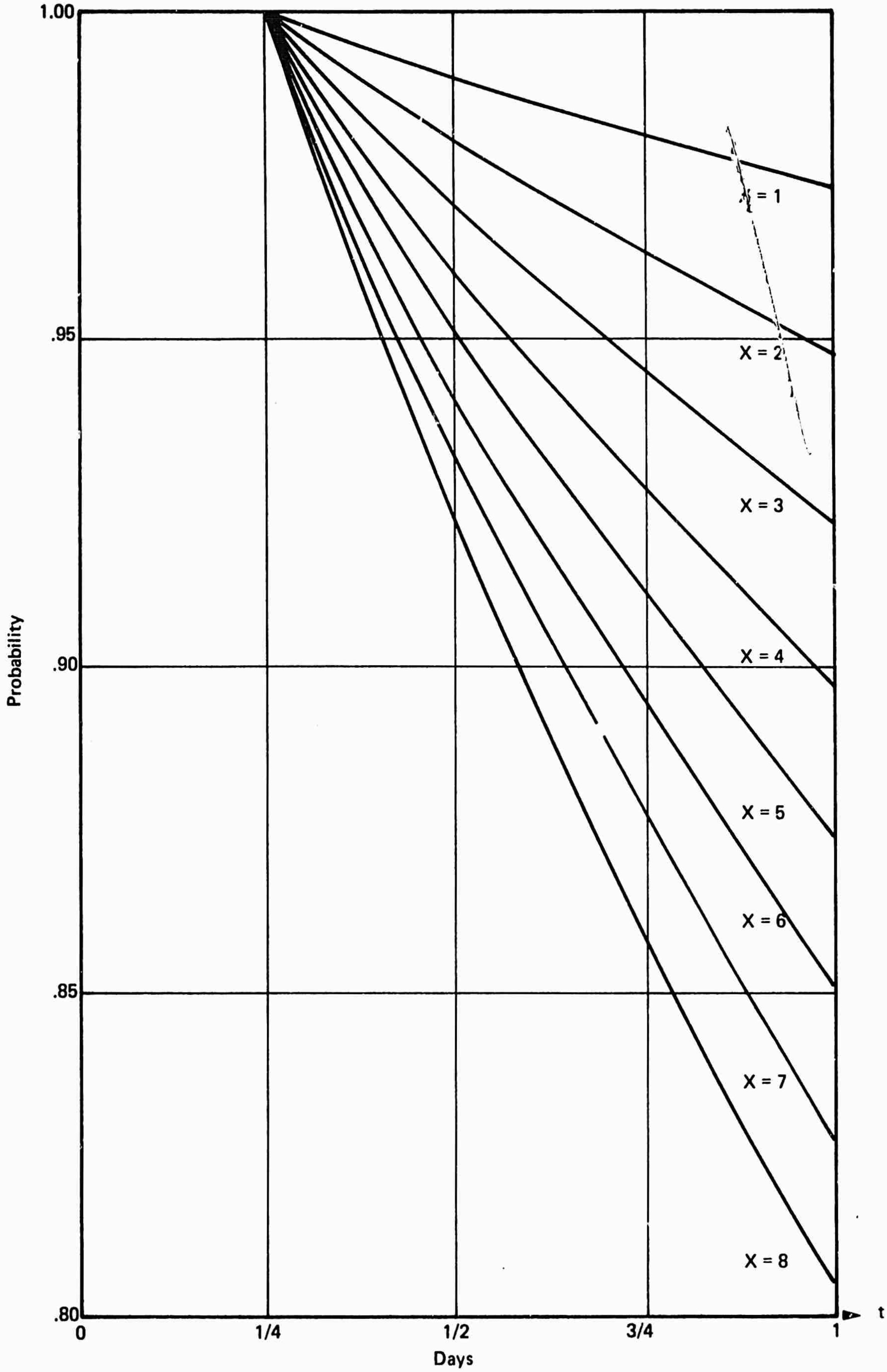

FIGURE 5. HVU SURVIVAL PROBABILITY AS A FUNCTION OF TIME (t) AND NUMBER (X) OF RED MISSILE CRUISERS

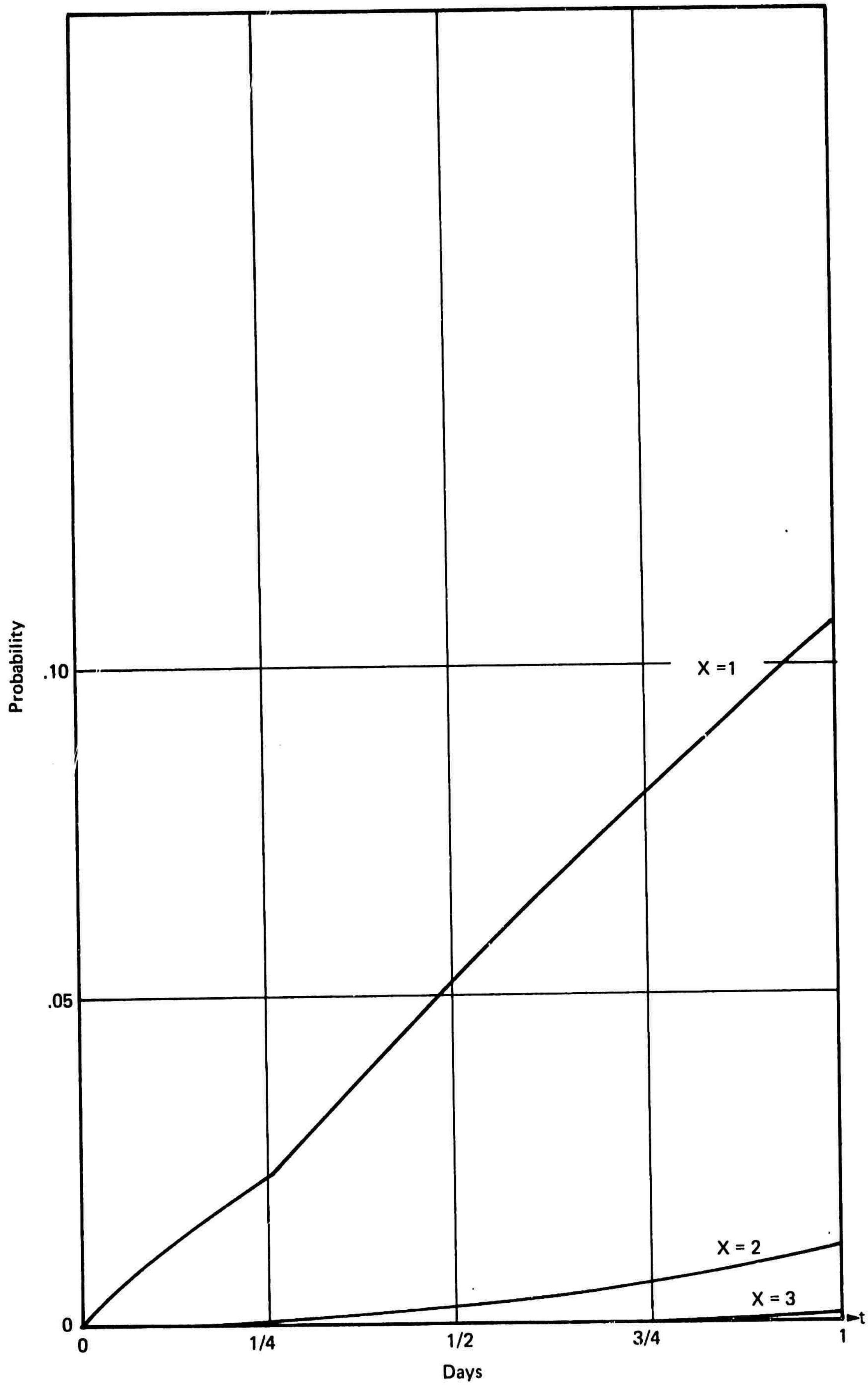

FIGURE 6. PROBABILITY THAT ALL X MISSILE CRUISERS HAVE BEEN DESTROYED OR EXHAUSTED AS A FUNCTION OF TIME t, FOR X = 1, 2, and 3

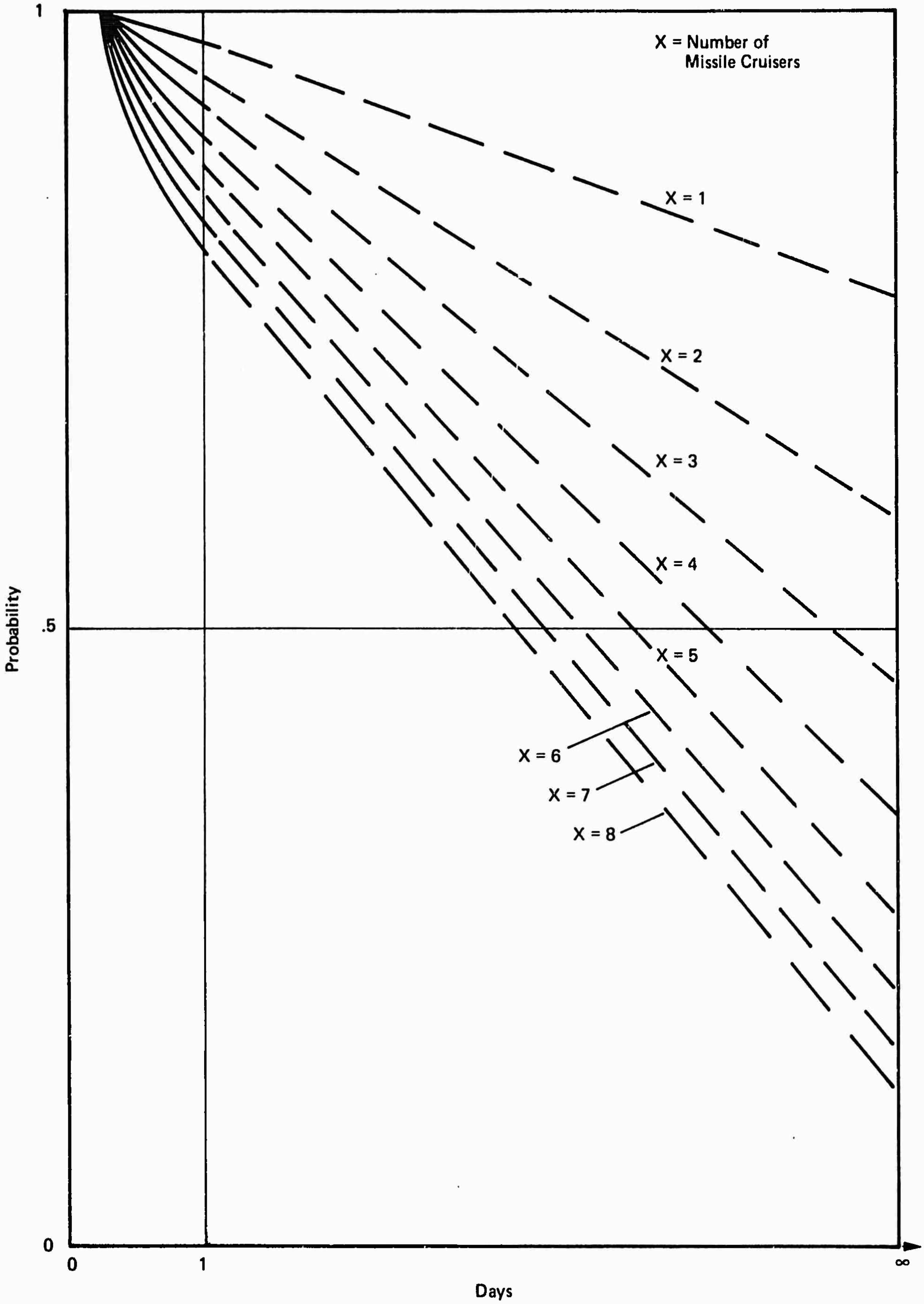

FIGURE 7. LIMITING PROBABILITIES OF HVU SURVIVAL AS TIME (t) BECOMES INFINITE

air threat. Key assumptions used in arriving at this conclusion are that the detectability of an individual RED missile cruiser raid group at the AEW barrier will not decrease as the size of the raid group is increased, and that a detected raid group will not be allowed to destroy the HVU. These and other assumptions are discussed in each of the appropriate submodel subsections.

For a numerical example exercising the overall surface model, which comprises the Air Barrier Detection Model, the Air/Surface Engagement Model, and the Search Model, suppose that just X = 4 RED missile cruisers are initially available. By the preceding result, the optimal formation into raid groups will be 1^4. Supposing the probability of detection at the AEW barrier to be $\epsilon \equiv .6$ independent of raid-group size, the probability of having each possible number 0, 1, 2, 3, or 4 missile cruisers pass through the barrier undetected is given in Table 2. Attrition to the VA suffered in destroying the detected raid groups can be computed as above for each possible number of detected groups, and (weighting each result with its probability of occurrence) the expected VA attrition thereby determined; these computations will be omitted here, since they do not affect the other model results. Finally, with the search model parameters defined as in the earlier numerical example, the probability of HVU survival to the end of the first day (say) can be determined as a function of number of undetected RED missile cruisers from Figure 5. Summarizing:

Number of Undetected Cruisers	Probability That this Number Will be Undetected	Approximate Probability of HUV Survival to End of First Day, Given this Number Cruisers
4	.0256	.90
3	.1536	.92
2	.3456	.95
1	.3456	.97
0	.1296	1.00

Combing these results, the probability of HVU survival through the first day, given an optimal RED strategy, is

$$(.0256)(.90)+(.1536)(.92)+(.3456)(.95)+(.3456)(.97)+.1296 = .96,$$

approximately.

AIR BARRIER DETECTION MODEL

The Air Barrier Detection Model is a modification of the Air Barrier Engagement Model described in Reference 3, p. 21 ff. The present notation is much the same as before, viz:

S = Number of sectors, each having the same sector width

p = Proportion of each sector covered by BLUE VA against RED missile cruisers

d = Spacing between RED missile cruiser raid groups

N = Number of RED raid groups

X = Total number of RED missile cruisers available

X_i = Number of RED missile cruisers in i^{th} raid group, for $i = 1, 2, \ldots, N$

$\epsilon(X_i)$ = Probability that BLUE AEW detects the i^{th} raid group, consisting of X_i missile cruisers, given that the raid group passes through the "covered" part of some sector.

In the present (missile cruiser) air barrier model, in contrast with the earlier (air reconnaissance) air barrier model, provision is made for the possibility that ϵ may depend on the size of the incoming RED (raid or recce) group. But, as before, the probability that BLUE detects a RED group entering a part of a sector that is not "covered" is zero.

The assumption (Reference 3, p.25) that the mid-point of the "covered" part of the sector is a random variable uniformly distributed over its possible locations will not be required in the present model. Instead, several simplifying assumptions are introduced that seem particularly appropriate in the missile cruiser case.

The first assumption is that, against the missile cruiser threat, each sector is uniformly "covered".

The "p = 1" Assumption

It is assumed that

$$p = 1. \tag{2.1}$$

This assumption would be inappropriate in the general air-reconnaissance case, since it would preclude the possibility of a RED recce group infiltrating through an AEW sector at a point beyond the range of the sensors of the patrolling AEW aircraft. Given the high speed of the recce aircraft, this possibility must be reckoned with whenever the sector width is more than a little greater than could be "covered" by an essentially stationary AEW aircraft. The relatively low speed of the missile cruisers, on the other hand, would seem to make it impossible for them to infiltrate without being within range of the AEW barrier sensors for at least some period of time. 1/

The second assumption is that the probability ϵ of detecting a RED missile cruiser group is independent of both the sector and the point within the sector where the raid group passes through the barrier.

Position-Independence of ϵ Assumption

It is assumed that ϵ does not depend on where a raid group crosses the AEW barrier. Clearly the validity of this assumption depends on the BLUE AEW barrier patrol strategy. The latter will probably be optimized, or nearly optimized, against the RED air threat, both because the air threat to the HVU appears to be greater than the surface threat and because the AEW barrier is one of the principle counters to the air threat, whereas for detecting missile cruiser raid groups other available means (such as sensors aboard BLUE submarines and LVUs, and the BLUE General Area Force) may be more efficacious. Thus the validity of assuming the position-independence of ϵ must ultimately be referred to the results of the combined model (subsuming air, surface, and submarine threats). The assumption is introduced provisionally at the present time, subject to a later sensitivity analysis when the combined model is completed.

Under the preceding two assumptions the spacing d between missile cruiser raid groups becomes irrelevant. So far as attempting to infiltrate missile cruisers undetected through the barrier is concerned, possible RED strategies reduce to determining how many raid groups to form and how many missile cruisers to put into each raid group. A third assumption follows.

1/ The BLUE deployment strategy for AEW barrier sensors does, however, determine the validity of the "p=1" assumption. Thus it will later be necessary to verify in the combined model that the "p=1" assumption remains valid. (See further remarks under the "Position-Independence of ϵ" assumption.)

Stochastic Independence of Raid Group Detections Assumption

The probability of detecting any given raid group is assumed to remain the same regardless of whether and what other raid groups have already been detected. This assumption has an analogue in the earlier Air Barrier Engagement Model, where it was assumed that the probability of detecting a recce group entering any given sector was independent of whether and what recce groups had already been detected in other sectors. It might be conjectured that in practice the probability of detecting the second and subsequent recce or missile cruiser raid groups would be greater than the probability of detecting the first one, owing to the greater state of alert after the first detection. BLUE communications policy would affect in an obvious way whether and to whom such an alert would be transmitted. Subsequent sensitivity analysis will be required to determine the extent to which such considerations affect the results in the present paper.

The present model takes account of the facts that raid groups must be formed by allocating so many missile cruisers to the first raid group, so many to the second, etc., an integer number of missile cruisers going into each raid group, until the supply of available missile cruisers is exhausted. Thus,

$$X_1 + X_2 + \dots + X_N = X \tag{2.2}$$

and $$X_i \text{ is a positive integer, for all } i. \tag{2.3}$$

For definiteness it will be assumed that the X_i are always so labelled that

$$X_1 \geq X_2 \geq \dots X_N. \tag{2.4}$$

Any set of integers $X_1, \dots, X_N$ satisfying (2.2) and (2.3) is called a partition of the integer X. Given X, the number of sets of integers $X_1, X_2, \dots, X_N$ satisfying (2.2) through (2.4) is the number of (unrestricted) partitions of X. The number of essentially different ways RED can allocate X missile cruisers into raid groups is therefore equal to the number of unrestricted partitions of X. A graph of this number as a function of X is given in Figure 3; values for X = 1, 2, ..., 10 are indicated by crosses, connected by dashed segments of straight lines, and the subsequent behavior of the function for $X \leq 30$ is suggested by a solid curve. Note that the scale on the left-hand side is logarithmic.

It is customary to denote particular partitions simply by writing their elements (the X_i) in descending sequence, using an exponent to denote repetition. Thus $4\ 2^2\ 1^3$ is the partition of 11 for which $N = 6$, $X_1 = 4$, $X_2 = X_3 = 2$, $X_4 = X_5 = X_6 = 1$. In the present paper this partition notation will be used to denote RED allocations of missile cruisers into raid groups.

When the number of available missile cruisers is small no computational difficulty is experienced in treating explicitly each possible allocation into raid groups. For larger numbers of missile cruisers such a straightforward computational approach may soon become unmanageable, however, as the growth of the solid curve in Figure 3 suggests. The present paper shows how to determine optimal strategies in the general case when X is small. When X is allowed to take on any finite positive integer value, no matter how large, a methodology is developed allowing optimal strategies to be developed for the surface model taken in isolation. Further research will be required, however, to determine optimal strategies for large X in the combined model— if, indeed, it turns out that values of X large enough to present computational difficulty are of interest in the combined model.

Applying the earlier assumptions, the probability that exactly n missile cruisers are detected at the AEW barrier is given by the coefficient of t^n in the expansion of

$$A(t) = \prod_i \left[1 - \epsilon(X_i)(1-t^{X_i})\right] \tag{2.5}$$

(For derivation, see Reference 4, p.251.) The expected number of missile cruisers to pass undetected through the barrier is then given by

$$\frac{d}{dt} A(t)\Big|_{t=1} . \tag{2.6}$$

Formulas for the variance and for still higher moments may be developed similarly. Formula (2.5) is all that will be required for the rest of the present section.

To be discussed later on are the consequences of a fourth possible assumption.

Monotonicity of $\epsilon(X_i)$ Assumption

ϵ may on occasion be assumed to be a nondecreasing function of X_i. The content of this assumption is that as the size of the raid group is increased, the probability of detecting the raid group at the AEW barrier does not grow less. Weak as this assumption may appear, it is sufficient to determine optimal RED allocation strategies under the conditions described and to be described in the surface defense model, taken in isolation.

When the surface defense model is combined with the other models, it may be necessary to assume more about $\epsilon(X_i)$ than is stated in the Monotonicity assumption if optimal RED and BLUE strategies are to be determined. In particular, when the surface defense model is combined with the air model, the increased detectability of larger RED missile cruiser raid groups must be traded off against their increased survivability against attack by the BLUE VA and against the increased casualties which they can inflict on the VA. Indeed, a situation can be conceived of in which the VA dare not attack a strong missile cruiser raid group for fear of becoming too depleted successfully to carry out the primary mission. This hypothetical situation suggests the sort of analysis for which the combined model will be required.

AIR/SURFACE ENGAGEMENT MODEL

Let X be the number of RED missile cruisers in a raid group and let this group be attacked by Y BLUE aircraft. Thus each cruiser can expect to encounter Y/X aircraft. Assuming that each cruiser has a probability P of surviving one attack, then the probability that it survives the attack by the Y/X aircraft is $P^{Y/X}$. Or if $P = \exp(-\alpha)$, then the survival probability is $\exp(-\alpha Y/X)$. Finally, the expected number of survivors out of the original X is given by $X \exp(-\alpha Y/X)$. This is an example of a so-called "numerically vulnerable threat".

When BLUE aircraft attack a RED missile cruiser raid group, both sides are assumed to be numerically vulnerable. Specifically, it is assumed that

$$X_i' = X_i \exp\left\{-\alpha Y_i/X_i\right\} \qquad (2.7)$$

$$Y_i' = Y_i \exp\left\{-\beta X_i/Y_i\right\} . \qquad (2.8)$$

Here the i^{th} raid group is attacked, and

X_i	is the number of RED missile cruisers in the raid group under attack
Y_i	is the number of BLUE aircraft attacking the raid group
X_i'	is the number of RED missile cruisers surviving the attack
Y_i'	is the number of BLUE aircraft surviving the attack
α	is a fixed positive parameter characterizing the susceptibility of RED missile cruisers to attack by BLUE aircraft
β	is a fixed positive parameter characterizing the susceptibility of attacking BLUE aircraft to defensive fire from the RED missile cruisers.

The assumption of numerical vulnerability on either side implies that either side will derive an advantage, both offensively and defensively, from concentrating its forces. From this point of view, RED prefers a single large missile cruiser raid group to several smaller ones.

BLUE can choose whether and when to attack each detected RED raid group, and how much of his available force to use in each attack. If his air resources are extremely limited, BLUE might elect to attack a missile cruiser raid group with the VA only if the position of the raid group put the HVU in imminent danger. RED, knowing or suspecting this policy on the part of BLUE, would then gain a reduction in any BLUE target ambiguity. On the other hand, given a superabundance of air resources, BLUE could still choose to attack a raid group only when its position put the HVU in imminent danger: with the HVU otherwise secure, whether or not BLUE attacks raid groups that are not immediately threatening depends on BLUE's secondary objectives.

Another possibility is that BLUE might withdraw the HVU rather than let it come within range of a detected missile cruiser raid group, and rather than use limited VA resources to attack the raid group, BLUE could adopt the same policy with BLUE's LVUs, thereby retaining target ambiguity. Although withdrawing the HVU might abort BLUE's mission, aborting the mission might be considered less serious than risking loss of the HVU to the missile cruiser raid group or, having depleted the VA by attacking the raid group, to the RED aircraft. Of course it is recognized that, at least conceptually, there are positions from which withdrawal is not possible.

In all cases a key question is, how many VA are necessary to use to reduce a missile cruiser raid group of given initial size to the point where it can no longer pose a threat to the HVU? Not just one raid, but rather a succession of raids, may be required to effect this reduction. The attrition suffered by the VA in this process is also of major interest.

A straightforward approach to answering this question is as follows. Suppose the initial size X_1 of the RED raid group is given. Let the initial size Y_1 of the BLUE VA contingent assigned to attack the raid group be the independent variable. Then the numbers X_1' of RED survivors and Y_1' of BLUE survivors are given by (2.7) and (2.8) as functions of Y_1. Supposing now that all BLUE survivors are used to attack the RED survivors in a second raid, Equations (2.7) and (2.8) (with an obvious change of notation) may be used to determine the numbers of RED and BLUE survivors of the second raid. The process may be further iterated any finite number of times. Since the numbers of RED and BLUE survivors are always reduced, but remain positive, after each iteration, these numbers must approach non-negative limits as the number of iterations is continued indefinitely. The limits themselves will be functions of Y_1. The limiting value of the number of RED survivors is a non-increasing function of Y_1, and this fact may be used to determine how large the initial value Y_1 must be in order, after enough iterations, to reduce the missile cruiser group to the point where it can no longer pose a threat to the HVU.

A method of estimating the limiting numbers of RED and BLUE survivors to within any desired degree of accuracy is given in Appendix A. There it is shown that the limiting number of RED survivors is zero if and only if $\alpha Y_1^a \geq \beta X_1^a$ and that the limiting number of BLUE survivors is zero if and only if $\alpha Y_1^a \leq \beta X_1^a$. In other words, at least one of the limiting numbers is zero, and both of the limiting numbers are zero only in the borderline case where $\alpha Y_1^a = \beta X_1^a$.

Even when only a small number of raids is contemplated, knowledge of these limits may be valuable. For, first of all, they show what cannot be accomplished: if BLUE cannot use the foregoing iterated-attack strategy to reduce the RED raid group to below a specified level in an infinite number of raids, then a fortiori BLUE cannot use it to do so in a finite number of raids. Secondly, convergence to the limiting values is frequently very rapid, and in such cases the limiting values may provide useful approximations to the actual force levels after just a few raids. And finally, knowledge of the limiting values may be used to determine at what point in the iterative use of (2.7) and (2.8) the limiting values have essentially been reached, rendering further computation unnecessary.

There is a caveat in connection with the foregoing iterated-attack strategy: it may not be optimal for BLUE. For instance, consider the following extreme example, chosen for ease in computation. Suppose that after the first raid the numbers of RED and BLUE survivors are 10,000 each. Suppose $\alpha = \beta = \ln 100$. If BLUE follows the foregoing iterated-attack strategy, then the numbers of RED and BLUE survivors will be 100 each after the second attack and 1 each after the third attack. On the other hand, if BLUE allocates only 5,000 of the first-raid survivors to the second raid, holding the remaining 5,000 first-raid survivors in reserve, then the number of RED second-raid survivors will be 1,000; the number of BLUE second-raid survivors will be .5 and these, added to the 5,000 first-raid survivors held in reserve, will yield a total BLUE force of 5,000.5 after the second raid. If, to continue, BLUE then allocates all 5,000.5 to the third raid, the number of RED third-raid survivors will be just less than 10^{-7}, while the number of BLUE third-raid survivors will be not quite 2,000. This example shows, incidentally, the tremendous advantage that can accrue to the player—BLUE, in this case—who can choose how much of his force to apply on each attack; the only difference between the players at the start of the second raid was that BLUE had this option, whereas RED was required to commit his raid group in its entirety in every BLUE attack.

Owing to the possibility that a situation like the one just discussed may develop, BLUE force level requirements based on an assumption that the foregoing iterated-attack strategy will be used will be conservative. They will be conservative because a BLUE strategy better than the assumed strategy may exist, and because use of an improved strategy would reduce BLUE's force level requirements. Further study is required to delimit the conditions under which the foregoing iterated-attack strategy is optimal and to determine, in cases where it is not optimal, what the optimal strategy is.

ALTERNATIVE AIR/SURFACE ENGAGEMENT MODEL

The Air/Surface Engagement Model discussed above deals in fractions of airplanes and missile cruisers and treats probabilistic outcomes in a sense as if they were deterministic. In some cases the quasi-deterministic outcomes thus obtained are the same as the expected values of the possible probabilistic outcomes, but in other cases, particularly when numbers that represent legitimate expected values are input to algorithms that treat them like deterministic values, the algorithmic outputs may represent expected values no longer. To overcome these objections, or to check in representative cases how much the algorithmic outputs differ from the expected values (or other process measures) with which they are associated, the following alternative model may be used.

Let $T((X_i, Y_i), (X_i', Y_i'))$ be the coefficient of $s^{X_i'} t^{Y_i'}$ in the expansion of

$$\left(1 - (1-s) \exp\left\{-\alpha Y_i/X_i\right\}\right)^{X_i} \left(1-(1-t) \exp\left\{-\beta X_i/Y_i\right\}\right)^{Y_i} \tag{2.9}$$

if $X_i' \epsilon \{0, 1, \ldots, X\}$ and $Y_i' \epsilon \{0, 1, \ldots, Y\}$, and let it be zero otherwise. Then $T((X_i, Y_i), (X_i', Y_i'))$ may be interpreted as the probability, given that Y_i BLUE aircraft attack a raid group of X_i RED missile cruisers, that Y_i' BLUE aircraft and X_i' RED missile cruisers will survive the attack. The number of BLUE and RED survivors are then random variables taking on non-negative integer values, and (2.7) and (2.8) are replaced by

$$\overline{X_i'} = X_i \exp\left\{-\alpha Y_i/X_i\right\} \tag{2.10}$$

and

$$\overline{Y_i'} = Y_i \exp\left\{-\beta X_i/Y_i\right\}, \tag{2.11}$$

respectively. Here $\overline{X_i}$ is the expected number of RED survivors and $\overline{Y_i}$ is the expected number of BLUE survivors. The similarity between (2.7) and (2.10), and between (2.8) and (2.11), provides a link between the two models.

Let initial RED and BLUE force levels X_i and Y_i be fixed. Let the matrix $U = ||u_{jk}||$ be defined by

$$u_{jk} = T((a, b), (c, d)) \Leftrightarrow (j = a(Y_i+1) + b \;\; \& \;\; k = c(Y_i+1) + d) \tag{2.12}$$

for all $\{(a, b), (c, d)\} \subseteq \{0, \ldots, X_i\} \times \{0, \ldots, Y_i\}$, where $T\ (a, b), (c, d)$ is defined as in the preceding paragraph. U is then the Markov transition matrix whose $(j, k)^{th}$ entry gives the probability of going in the course of an attack from state (a, b) to state (c, d) — i.e., from the state where just a missile cruisers and b aircraft are operational to the state where just c missile cruisers and d aircraft are operational. Here (a, b) and (c, d) are related to j and k as on the right-hand side of (2.12).

In the Markov chain having U as transition matrix a state (a, b) is an absorbtion state if and only if a and/or b is zero. All other states are transition states and in fact "never return" states, in that the system, once having passed out of such a state, can never again return to it. The probability, after n attacks, that the system is in state (c, d) is given by entry k in the last row of the n^{th} power U^n of U; here the correspondence between (c, d) and k is as before. Means of determining the limiting probability distribution as n becomes infinite are well known. (See, for instance, Reference 5, p.404 ff.)

SEARCH MODEL

The Search Model is based in large part on Reference 2, Appendix A. Differences will be noted where appropriate.

In the present model it will be assumed that missile cruiser raid groups detected at the AEW barrier will not be permitted to approach within striking range of the HVU: either they will first be destroyed by the VA or by some other means, or else the HVU will withdraw in time to prevent their approach. The situation in which this assumption is not made will be reserved for later study.

Accordingly the search model will deal only with those missile cruisers that are undetected at the AEW barrier. A second assumption used in the present model is that, although they may have crossed the AEW barrier in groups some of which contained more than one cruiser (thereby gaining the benefit of mutual support in the event of detection and attack by the VA), all missile cruisers that are undetected at the barrier will split off from their groups and search for the HVU on an individual basis. How will RED know for sure which cruisers are undetected at this point? He will not, especially if (in line with the earlier discussion) BLUE uses the VA only to attack those missile cruiser groups that imminently threaten the HVU. By crediting RED with knowledge that in fact he may not have, the model will result in an analysis that is more pessimistic for BLUE than is proper. Decisions about how BLUE should act based on this analysis might therefore be expected to err on the side of conservatism. A sensitivity analysis will be required to determine the extent to which this assumption affects the model results.

By searching for the HVU individually rather than in groups, the missile cruisers increase the probability of encountering the HVU within any specified period of time. This increase in probability acts in RED's favor. Likewise upon encountering an LVU or HVU the probability of the raid group being detected is smaller when it consists of just one cruiser. The latter decrease in probability also acts in RED's favor. But in an exchange of fire with an LVU or HVU, the smaller raid group has less chance of destroying its adversary and less chance of itself surviving, in whole or in part, than does the larger raid group. These last considerations act in BLUE's favor. In many

cases it may be that the considerations just listed as acting in RED's favor outweigh the considerations acting in BLUE's favor, so that searching for the HVU with individual missile cruisers rather than with groups is really the optimal RED strategy. Here it is simply assumed that RED will use the strategy of individual search. But an upper bound on the extent to which this assumption may warp the model results is readily had: repeat the analysis, crediting the individual RED cruisers with the firepower appropriate to larger missile cruiser raid groups. The results of the repeated analysis are guaranteed to be optimistic for RED: and if they do not differ too much from the results of the first analysis, then the strategy of searching with individual cruisers cannot differ too much in its consequences from the optimal RED strategy, whatever that may be.

Suppose that X RED missile cruisers have penetrated the AEW barrier undetected. These cruisers begin to search for the HVU, each cruiser searching randomly and independently of the other cruisers.

In the course of its search each cruiser may encounter the HVU, one of I types of BLUE LVU, or the BLUE general area force. The number of encounters of any one of these types in a search path of fixed length is assumed to be a Poisson-distributed random variable, and each such random variable is assumed to be stochastically independent of every other one. In particular, the probability of encountering the HVU in a short length ds (nautical miles) of search path is assumed to be λ_0 ds; of encountering an LVU of type i, λ_i ds for $i = 1, 2, \ldots, I$; and of encountering the general area force, (θ/U) ds, where θ is the probability of encountering the general area force in a short period dt (days) of time and U is the assumed speed of the searching cruiser (in nautical miles per day). It is assumed that no encounters of any type are possible except when the cruiser is under way.

An encounter with the HVU may result in the HVU killing the cruiser (probability $1 - \sigma_0 = k_0$) or in the cruiser killing the HVU (probability $\sigma_0 \delta_0$). An encounter with an LVU or type i may result in the LVU killing the cruiser (probability $1 - \sigma_i$) or in the cruiser firing its missiles at the LVU (probability $\sigma_i(1 - \delta_i)$), thereby becoming "exhausted" and having no weapons left to fire in any subsequent encounter with the HVU. An encounter with the general area force may result in the cruiser being killed (probability $1 - \sigma_a = K_a$). An encounter having any of the results just enumerated will be called an "encounter*" with asterisk. Since an encounter may result in neither HVU nor cruiser being destroyed, and in the cruiser not becoming exhausted, not every encounter is an encounter*, even though every encounter* is an encounter.

A given missile cruiser destroys the HVU only if it destroys the HVU on its first encounter*. Therefore the probability that it destroys the HVU between path lengths s and s + ds is the product of the probability

$$b(s) = \exp\left\{-\left(\sum_{i=1}^{I}\lambda K + (\theta/U)K + \lambda_0 E_0\right)s\right\} \qquad (2.13)$$

that the cruiser has no encounter* during the first s nautical miles of search path, and the probability $\lambda_0\sigma_0\delta_0$ ds, conditioned on the cruiser having had no previous encounter*, that the cruiser then kills the HVU within a small length ds of search path. (The parameter E_0 in (2.13) is defined by $E_0 = 1 - \alpha_0 + \alpha_0\delta_0$.) Integrating this product over the path length from 0 to L yields the probability

$$B(L) = \lambda_0\delta_0\sigma_0(1-b(L))/\ln b(-1) \qquad (2.14)$$

that the cruiser kills the HVU within the first L nautical miles of the cruiser's search path. Equation (2.14) agrees with the results of Reference 2, Appendix A, although the present method of proof is slightly different.

The present model is essentially the same as the model of Reference 2, Appendix A. The latter model takes account of the fact that the searching vessels (submarines, in the previous model, as opposed to surface vessels in the present model) must spend time classifying any LVU or HVU encountered. The time required for classification is h_i days for an LVU of type i and h_0 days for the HVU.[2] The time t required to search a path of length s depends not only on the speed U of the searcher, but also on the number n_i of encounters with

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2/ Although these times represent average times they are treated in the model as though they were time delays of fixed, determinate length. Strictly speaking, this treatment may, except in special cases, be illogical: if the HVU takes longer to classify than any LVU, for instance, then as soon as more time has been spent attempting to classify an object than would be required for any LVU, it may be concluded that the object is an HVU. But then the HVU takes no longer to classify than some LVU—i.e., $h_0 = \max h_i$—and this contradicts the assumption that $h_0 > \max h_i$. But this illogicality is not pernicious, provided it can be shown that the model results are, to within an acceptable degree of approximation, the same as would be obtained treating the time to classify each object as a random variable with appropriate probability distribution. A sensitivity analysis is required to make this determination.

In the present case such a sensitivity analysis can be made using the same sort of model, but re-interpreting some of the parameters. For instance, double subscripts may be introduced and λ_{ij} ds defined as the probability of encountering a "type i" LVU in a small length ds nautical miles of search path, and of requiring h_{ij} days to classify it. For each i, h_{ij} ranges over a finite or countably infinite set of values indexed by j. The probability distribution of the time required to classify the HVU is introduced similarly.

"type i" LVUs, for each i, and the number m of encounters with the HVU during the search. Moreover, none of the encounters may have been an encounter*, since otherwise the search would not have lasted so long. The number of encounters with a "type i" LVU, that are not encounters*, in a search path of length s is Poisson-distributed with mean $(1 - K_i)s$, for each i, and the number of encounters with the HVU that are not encounters* is Poisson-distributed with mean $(1 - E_0)s$. Using the assumption of independence, the probability $C(n_1, \ldots, n_I, m|s)$ that in a given search path of length s there will have been just n_1 encounters (that were not encounters*) with some "type 1" LVU, ..., n_I encounters (that were not encounters*) with some "type I" LVU, and m encounters (that were not encounters*) with the HVU, is therefore given by

$$C(\underline{n}, m|s) = ((1-E_0)s)^m \exp(-(m+\Sigma_{n_j})s)\,\pi\,((1-K_i)s)^{n}/n_i!\,m! \qquad (2.15)$$

when the sum is over j and the product is over i, (n, j) = 1, 2, ..., I, and $\underline{n}$ is the vector with components $n_1, n_2, \ldots, n_I$.

Combining with the results of the preceding paragraph, the probability that a given missile cruiser will within t days have an encounter* resulting in HVU destruction is

$$1 - v(t|1, \underline{\lambda}, \lambda_0, \theta) =$$

$$= \lambda_0 \delta_0 \sigma_0 \Sigma_{\underline{n}, m} \int_0^{L(\underline{n},m,t)} C(\underline{n}, m|s) \exp\left(-(\lambda_0 E_0 + \theta/UK_a + \Sigma \lambda_i K_i)s\right) ds \qquad (2.16)$$

when $$L(\underline{n}, m, t) = \max\left((t - mh_0 - \Sigma n_i h_i)U, 0\right). \qquad (2.17)$$

It may be most reasonable to take the time of HVU destruction as occurring h_0 days after the beginning of the encounter* resulting in HVU destruction, since the cruiser must (correctly) classify the HVU prior to launching its missiles. Similarly the time at which a cruiser becomes "exhausted" against a "type i" LVU might be taken as h_i days after the start of that encounter*; in this case the cruiser's classification is incorrect, but is presumably made as slowly or as quickly as a correct classification would have been made. On the other hand in an encounter* where the missile cruiser itself is destroyed, the cruiser's classification process may not even have begun; and so it appears reasonable to take missile cruiser destruction as occurring at the start of such an encounter*. These conventions will be used in the present paper.

Note that according to (2.15), $C(\underline{n},m|s)$ is independent of θ. The need for such independence may be seen by observing that $C(\underline{n},m|s)$ is the probability of having the prescribed numbers of encounters (n_1 encounters with "type 1" LVUs, etc.) during an initial search path of length s nautical miles, <u>given that the cruiser survives such a search path with no encounter*</u>. Although θ must be used (as in (2.13) in computing the probability of the given event, its use in computing the probability of the event conditioned on the given event does not come into question. Equation (2.7) of Reference 2, p.A-11 appears to be in error in this respect. For this reason, and for a second reason about to be given, (2.16) differs somewhat from the analogous expressions to be found in the earlier work.

The second reason for (2.16) differing from the previous results is that in the earlier work (e.g., on page A-9) the expected time (given no encounter*) required to search a path of finite length is determined as a function of path length. The function inverse to this function is then determined, and used as if it gave the length of path searched as a (determinate) function of the length of time spent searching it. Although this procedure gives excellent approximations in some cases, it must be used circumspectly in any given application, until its validity as an approximation has been established for that application. The expression (2.16) is exact, but the expressions derived by the earlier method may be preferred in cases where they can be shown to be sufficiently accurate, since they are less cumbersome. Reference 2 treats the searching submarine as if it were immune to encounters with the general area force during the classification process. The same treatment is given the searching missile cruiser in the present model, though with perhaps less justification. For, the submarine would presumably be submerged during the classification process, as quiet as possible and with such noise as did emanate being masked by the target being classified. The cruiser, on the other hand, would be on the surface and therefore subject, in particular, to visual and radar detection. For these very reasons the cruiser classification times h_i and h_o might in practice be extremely short, with a consequent high probability of misclassification; if so, the question of whether or not the cruiser is subject to attack by the general area force while classifying would become academic, since (with high probability) the proportion of time spent classifying would be very slight. If, however, it is desired to have the model reflect a state of affairs in which the cruiser is subject to attack by the general area force while classifying, it is sufficient to change (2.16) by multiplying the integrand by

$$\exp\left\{-\theta K_a\left(mh_o+\sum_i n_i h_i\right)\right\} . \qquad (2.18)$$

From (2.16), the probability that the HVU survives destruction by any given missile cruiser, acting alone, for at least t days is $v(t|1,\lambda_1, \ldots, \lambda_I, \lambda_0, \theta)$. Since all X missile cruisers act independently of one another, the probability that the HVU survives the X missile cruisers for at least t days is this quantity raised to the X^{th} power, viz

$$\left[v(t|1,\lambda_1, \ldots, \lambda_I, \lambda_0, \theta)\right]^X . \qquad (2.19)$$

Expression (2.19) completes the development of the Search Model.

COMBINATION OF SUBMODELS AND OPTIMIZATION

The Surface Model is completed by combining the submodels into a single model and determining optimal RED and BLUE strategies in the combined model. In the present case BLUE's choice of strategy—e.g., the number and types of LVUs, the strength and placement of the AEW barrier, etc.—have been assumed to have taken place outside the boundaries of the Surface Model itself, so that there are no alternatives open to BLUE within the structure of the model. (The model is nonetheless useful to BLUE in making strategic decisions, like the ones listed above, outside the structure of the model. For, the effects of these decisions enter into the model as inputs, and the model results are therefore a measure of how good the strategy is that leads to these decisions. The model can thus be used to evaluate alternative BLUE strategies, even though the possible alternatives are not described within the model.)

Under the assumptions discussed above, the RED missile cruisers will be formed into raid groups that try to penetrate the BLUE AEW barrier undetected. Once past the AEW barrier, those raid groups that have successfully remained undetected will split up and search for the HVU on an individual basis. Raid groups that have been detected at the AEW barrier will not be permitted to approach the HVU; they will, at BLUE's discretion, be attritted by the VA, and in the event of extreme necessity BLUE will withdraw the HVU rather than allow them to draw too near. Since RED's main objective is to find and destroy the HVU, the raid groups detected at the AEW barrier will be considered only incidentally in the present model; they will be of greater concern in a later model combining air, surface, and submarine models, where the tradeoff between suffering VA attrition by attacking the detected raid groups, and possibly losing or having to withdraw the HVU on account of not attacking them, will be dealt with explicitly. How the undetected missile cruisers seek and attempt to destroy the HVU is discussed in connection with the Search Model, above.

Let X' be the number of missile cruisers crossing the AEW barrier undetected. Then X' is a random variable with probability distribution given by (2.5). By (2.19), the probability that RED fails to destroy the HVU within t days is

$$\left[v(t \mid 1, \lambda_1, \ldots, \lambda_I, \lambda_0, \theta)\right]^{X'} . \qquad (2.20)$$

Assume (to avoid triviality) that the expression in square brackets in (2.20) lies between 0 and 1, exclusive of endpoints. Then the expected value of (2.20) is minimized if and only if the initial RED allocation is 1^X, provided that $0<\epsilon(1)<1$ and that $\epsilon(X_i)$ is a non-decreasing function of X_i (the Monotonicity of $\epsilon(X_i)$ Assumption discussed above). This result is proved in Appendix B. Note that the allocation 1^X, which does not depend on t, minimizes the expected value of (2.20) for all t and hence for the limiting value as t approaches infinity, as well.

When the Monotonicity of ϵ Assumption holds, the RED allocation 1^X is optimal in the sense of minimizing the expected value of (2.20). When this assumption does not hold, some other initial allocation may be optimal in this sense. If X is not large, the optimal initial allocation may be determined directly from (2.5) and (2.20). For larger values of X, it seems that something more would have to be known or assumed about the function ϵ for useful conclusions to be able to be drawn.

III. COMBINED SURFACE AND SUBSURFACE MODEL

INTRODUCTION

The following two sections describe procedures for obtaining a mathematical description of the progress and outcomes of two types of attack on a carrier task group (CTG). The first of these is an attack by enemy surface and subsurface units which operate independently and attempt to destroy a target immediately upon identifying it as a preferred objective. The second type of attack is a coordinated air and sea attack in which enemy surface and subsurface units attempt to locate and hold contact on preferred targets of the CTG, delaying their attack so as to have it coincide with an air attack which is to occur at a predetermined time. The mathematical model for the first of these two types of attack, i.e., the surface and subsurface model, is presented in this section. The model for the air-sea type of attack is described in Section IV. Examples illustrating the use of the mathematical models for each of these types of attack, and the results that can be obtained, are also presented.

THE NATURE OF THE BATTLE

The combined surface and subsurface model treats the battle between Red forces comprised of missile-cruiser groups and submarines which have penetrated the air barrier around the Blue CTG, and the CTG, which consists of high value units (HVUs) and low value units (LVUs). During the battle, the missile-cruiser groups, of possibly different compositions, and individual submarines each independently search the area in which the CTG is located.

The Red objective is assumed to be the destruction of all HVUs of the Blue force. Moreover, the result of a decision by a Red unit to attack a Blue unit that it has encountered is to exhaust the unit's ordnance and thereby remove itself from further participation in the battle. Considering this fact and Red's battle objective, an effort by Red units to properly classify Blue

units before attacking them is essential, and to this end a Red unit, upon encountering a Blue unit, is assumed to suspend its search for a period required to determine if the unit is an HVU or an LVU. The required identification period in this model is taken to be a constant for a particular type of encounter, but to depend upon the nature of the encounter, i.e., the types of Red and Blue units involved.

The defense of the Blue force is accomplished by the local defenses of each unit against those Red units which it encounters, and by general area forces (GAF) which are able to attack, with impunity and constant effectiveness, all Red units throughout the duration of the battle. The defenses of individual Blue units are assumed to be inexhaustible.

DEFINITIONS AND ANALYTICAL APPROACH

An encounter is defined as any meeting between a Red unit and a Blue unit. An encounter which results in the removal of a unit from the battle, by either the destruction or ordnance exhaustion for a Red unit, or by destruction only for a Blue unit, will be referred to as an "engagement" (corresponding to encounter* of the previous section), a somewhat specialized usage of this term which excludes unsuccessful attempts by Blue forces to destroy a Red unit. The time of an engagement is taken to be the time at which the encounter occurs which leads to the removal, within the required Red identification period, of a unit.

The progress of the battle may be represented by states of the combined Red and Blue forces which are defined in terms of the numbers of units with distinguishable characteristics that have been destroyed. Transitions from one state to another occur only as the result of an engagement, as defined in the preceding paragraph. The existence of multiple transition possibilities from a given state of the system suggest that the battle action might be represented by a Markov chain proceeding from an initial state wherein no units of either side are inoperative, to one of a number of final, or "absorption" states. Absorption states consist of those states in which all HVUs have been destroyed and from which no further transitions are of interest, and those states in which all Red units have been destroyed and from which no further transitions are possible. Because of the irreversible nature of attrition, the transition states to these absorption states are "never return" states and all transition possibilities are, at most, one-way. Moreover, since each state requires a unique number of engagements for its occurrance, it can be characterized by a unique "step number" in the chain.

In order for the battle to be describable by a Markov chain, it is necessary that the processes involved be Markovian. However, this requires only that the transition probability from the present state of the system to another state be independent of the path by which the current state was reached, and under the assumptions specified earlier this requirements is fulfilled. Transition rates between states that are independent of time are

not essential, but such independence simplifies not only the calculation of transition probabilities, and hence the comparative likelihood of the Markov processes terminating in a particular absorption state, but also the computation of the probability that the system is in a given state at a specified time. For the engagements of the battle described above, the transition rates approach constants only asymptotically and can be taken as constants for finite times only as an approximation. The exact calculation of these transition rates and the relation of th exact values to the asymptotic limits are described in Appendix C.

In the following paragraphs, a simple battle will be considered for the purpose of illustrating the calculations that are required for the general case. The outcomes at various times will be calculated on the assumption that the approximation resulting from the use of asymptotic transition rates rather than their exact values is acceptable.

AN EXAMPLE

States of the System

Consider a Red force consisting of two undetected missile-cruiser groups of different composition, and two undetected submarines which are in all essential respects indistinguishable. This Red force searches an area containing a Blue force of one LVU and one HVU, intent upon destroying the latter. Let the index ℓ denote the type of Red unit and the index i the type of Blue unit, as follows:

ℓ = 1: missile-cruiser group 1

ℓ = 2: missile-cruiser group 2

ℓ = 3: submarines

i = 0: HVU

i = 1: LVU.

The states of the system can be represented by the number

$$n_{\ell=3}\ n_{\ell=1}\ n_{\ell=2}\ n_{i=1}\ n_{i=0},$$

where n_u is the number of units of type u that have become inoperative. Clearly, the initial state is the state 0 0 0 0 0. A complete listing, numbered by the index s, of all possible states S_s is shown in Table 3. The states are so ordered so that transitions $S_a \rightarrow S_b$ are impossible if a > b. States marked with an asterisk are inaccessible from state 1 and need not be considered further. A flow diagram of the Markov chain is shown in Figure 8.

TABLE 3

POSSIBLE STATES OF THE SYSTEM

			S_s		
s	ℓ_S	ℓ_1	ℓ_0	i_1	i_0
1	0	0	0	0	0
2	1	0	0	0	0
3	0	1	0	0	0
4	0	0	1	0	0
5*	0	0	0	1	0
6*	0	0	0	0	1
7	2	0	0	0	0
8	1	1	0	0	0
9	1	0	1	0	0
10	0	1	1	0	0
11	1	0	0	1	0
12	0	1	0	1	0
13	0	0	1	1	0
14	1	0	0	0	1
15	0	1	0	0	1
16	0	0	1	0	1
17*	0	0	0	1	1
18	2	1	0	0	0
19	2	0	1	0	0
20	1	1	1	0	0
21	2	0	0	1	0
22	1	1	0	1	0
23	1	0	1	1	0
24	0	1	1	1	0
25	2	0	0	0	1
26	1	1	0	0	1
27	1	0	1	0	1
28	0	1	1	0	1
29*	1	0	0	1	1
30*	0	1	0	1	1
31*	0	0	1	1	1
32	2	1	1	0	0
33	2	1	0	1	0
34	2	0	1	1	0
35	1	1	1	1	0
36	2	1	0	0	1
37	2	0	1	0	1
38	1	1	1	0	1
39	2	0	0	1	1
40	1	1	0	1	1
41	1	0	1	1	1
42	0	1	1	1	1
43	2	1	1	1	0
44	2	1	1	0	1
45	2	1	0	1	1
46	2	0	1	1	1
47	1	1	1	1	1
48	2	1	1	1	1

* Inaccessible from state 1.

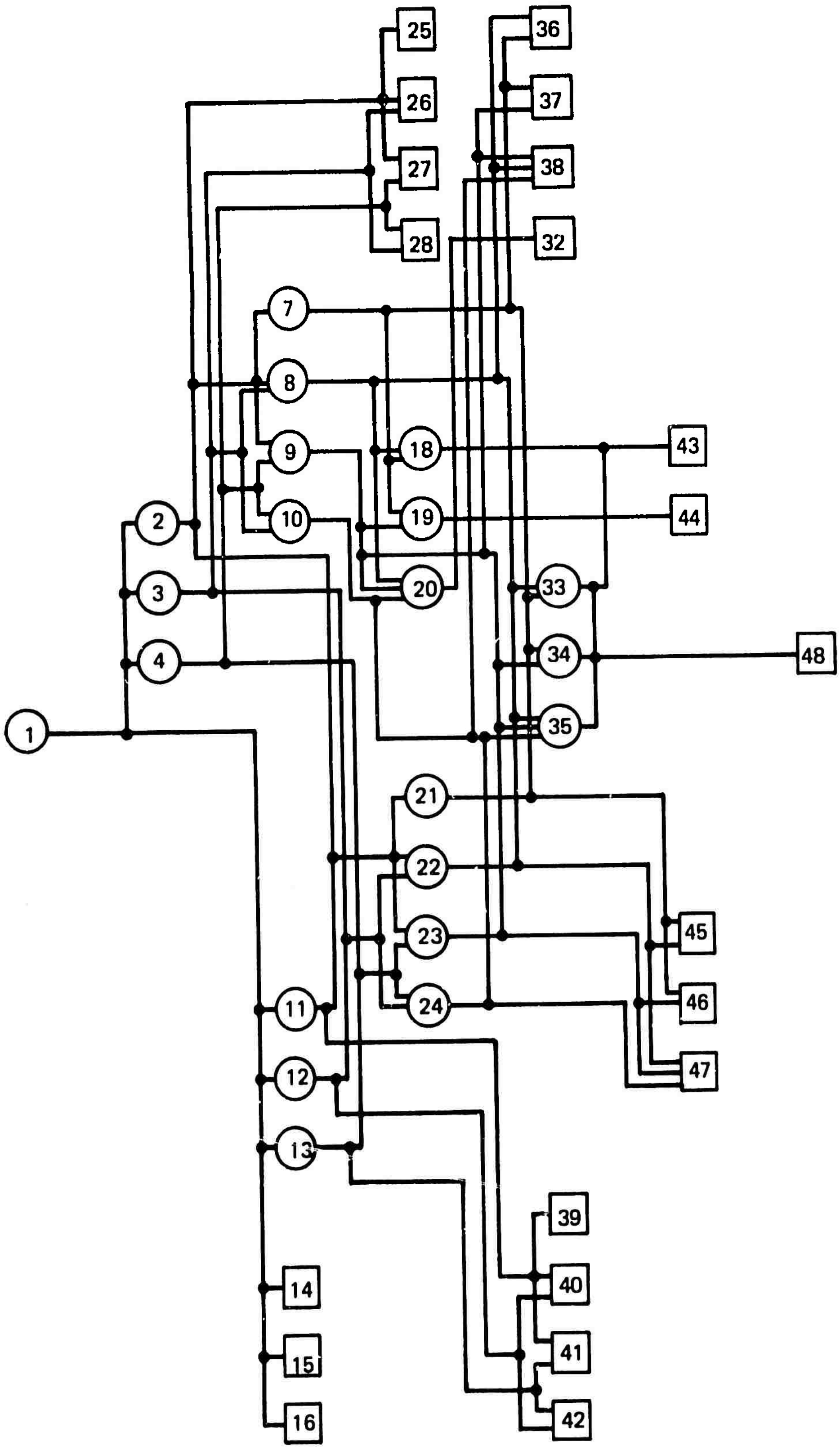

FIGURE 8. FLOW DIAGRAM OF MARKOV CHAIN

All flow except that from right to left is allowed. Absorption states are indicated by square boxes. A simplified flow diagram which indicates all allowed transitions (but does not seek to designate all unallowed transitions as such), and which clearly shows the step number of each state is shown in Figure 9. It is obvious in this figure that, at most, four transitions are required to reach an absorption state.

Figure 10 shows a skeletal transition matrix. States inaccessible from state 1 are designated by shaded vertical bands, indicating that transitions to these states need not be considered. Absorption states are designated by shaded horizontal bands, indicating that transitions from these states can be ignored. States between which direct transitions can occur and for which transition rates must be determined are marked by an x.

Time Dependent Behavior

If a constant transition rate λ_{ij} from state i to state j is assumed, the probability that the system, given that it is in state i, makes a transition to state j in time dt is $\lambda_{ij}\,dt$. The probability that it makes a transition to any state from state i in time dt is $\lambda_i dt$, where, $\lambda_i = \sum_j \lambda_{ij}$. Thus, the probability that the system, initially in state i, has not made a transition from state i in time t is $e^{-\lambda_i t}$. The probability that the system has made a transition from state i to state j in time t is

$$\frac{\lambda_{ij}}{\lambda_i}(1-e^{-\lambda_i t}).$$

The general expression for the probability that a system is in state k at time t is

$$P_k(t) = \sum_j \frac{\lambda_{jk}}{\lambda_k}\int_0^t \lambda_j P_j(t_j)\, e^{-\lambda_k (t-t_j)}\, dt_j . \tag{3.1}$$

This expression is simply the joint probability that the system has, at some time $t_j < t$, been in a state j from which a transition to state k can be made, has made the transition $j \rightarrow k$, and has not left the state k in the remaining period $t - t_j$. For a sequence of states 1, i, j, k and ℓ, the expression for the probability of occupancy of states with various step numbers can be obtained by the recursive use of Equation (3.1). Thus

$$P_i(t) = \frac{\lambda_{1i}}{\lambda_1}\int_0^t \lambda_1 P_1(t_1)e^{-\lambda_1 (t-t_1)}\, dt_1 = \frac{\lambda_{1i}}{\lambda_1}\left[\frac{e^{-\lambda_i t}}{(\lambda_1-\lambda_i)} + \frac{e^{-\lambda_1 t}}{(\lambda_i-\lambda_1)}\right] . \tag{3.1a}$$

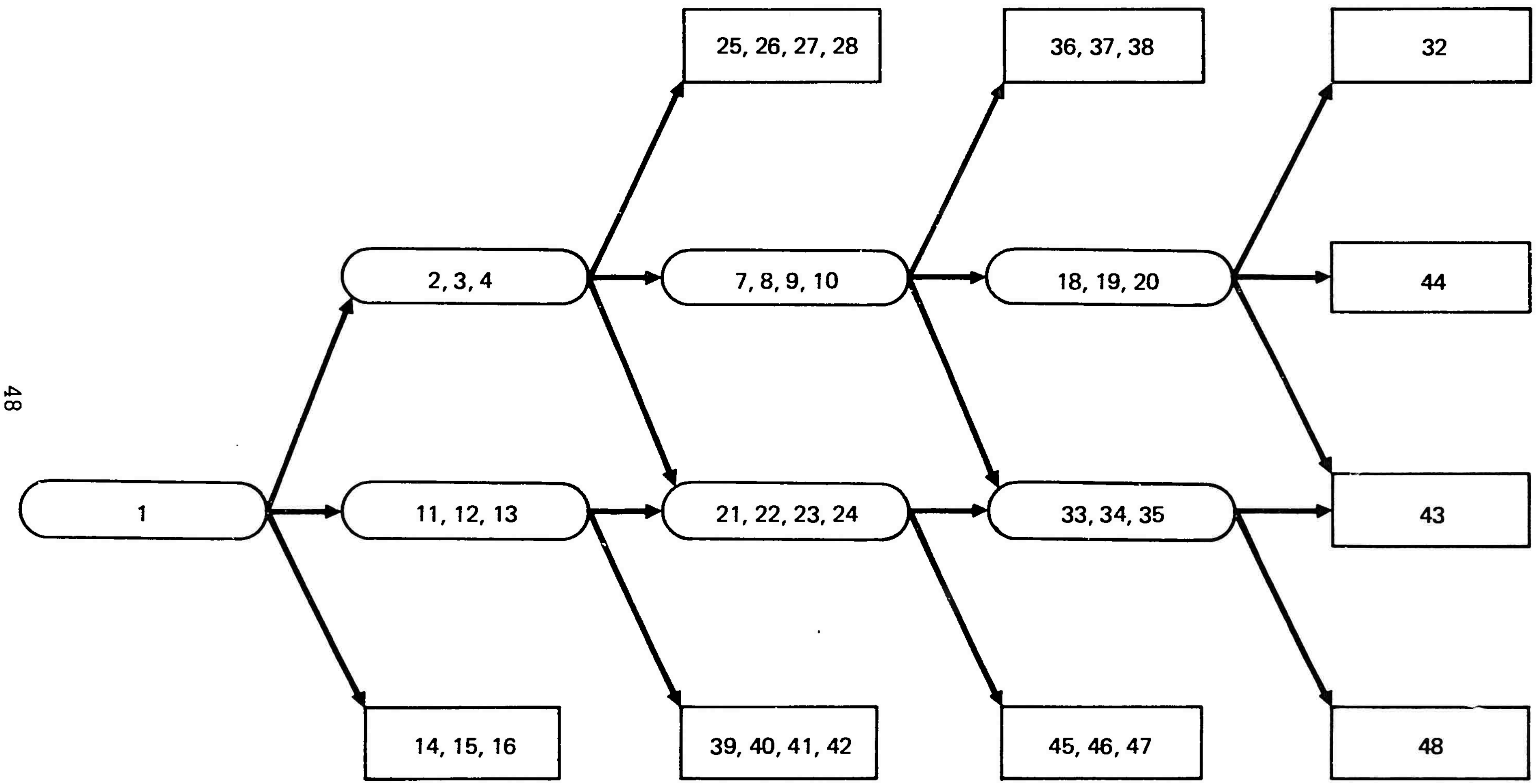

FIGURE 9. SIMPLIFIED FLOW DIAGRAM

TO

FROM	1	2	3	4	5•	6•	7	8	9	10	11	12	13	14	15	16	17•	18	19	20	21	22	23	24	25	26	27	28	29•	30•	31•	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48
1		X	X	X							X	X	X	X	X	X																																
2							X	X	X												X	X	X		X	X	X																					
3								X		X												X		X		X		X																				
4									X	X													X	X			X	X																				
•5																																																
•6																																																
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FIGURE 10. SKELETAL TRANSITION MATRIX

$$P_j(t) = \sum_i \frac{\lambda_{ij}}{\lambda_i} \int_0^t \lambda_i P_i(t_i) e^{-\lambda_i (t-t_i)} dt_i$$

$$= \sum_i \frac{\lambda_{1i}}{\lambda_1} \frac{\lambda_{ij}}{\lambda_i} \lambda_1 \lambda_i \left[\frac{e^{-\lambda_j t}}{(\lambda_i-\lambda_j)(\lambda_1-\lambda_j)} + \frac{e^{-\lambda_i t}}{(\lambda_j-\lambda_i)(\lambda_1-\lambda_i)} + \frac{e^{-\lambda_1 t}}{(\lambda_j-\lambda_1)(\lambda_i-\lambda_1)} \right]. \quad (3.1b)$$

$$P_k(t) = \sum_j \frac{\lambda_{ik}}{\lambda_j} \int_0^t \lambda_j P_j(t_j) e^{-\lambda_j (t-t_j)} dt_j = \sum_j \sum_i \frac{\lambda_{1j}}{\lambda_1} \frac{\lambda_{ij}}{\lambda_i} \frac{\lambda_{jk}}{\lambda_j} \lambda_1 \lambda_i \lambda_j \, .$$

$$\left[\frac{e^{-\lambda_k t}}{(\lambda_j-\lambda_k)(\lambda_i-\lambda_k)(\lambda_1-\lambda_k)} + \frac{e^{-\lambda_j t}}{(\lambda_k-\lambda_j)(\lambda_i-\lambda_j)(\lambda_1-\lambda_j)} \right.$$

$$\left. \frac{e^{-\lambda_i t}}{(\lambda_k-\lambda_i)(\lambda_j-\lambda_i)(\lambda_1-\lambda_i)} + \frac{e^{-\lambda_1 t}}{(\lambda_k-\lambda_1)(\lambda_j-\lambda_1)(\lambda_i-\lambda_1)} \right]. \quad (3.1c)$$

$$P_\ell(t) = \sum_k \frac{\lambda_{k\ell}}{\lambda_k} \int_0^t \lambda_k P_k(t_k) e^{-\lambda_k (t-t_k)} dt_k = \sum_k \sum_j \sum_i \frac{\lambda_{1i}}{\lambda_1} \frac{\lambda_{ij}}{\lambda_i} \frac{\lambda_{jk}}{\lambda_j} \frac{\lambda_{k\ell}}{\lambda_k} \, .$$

$$\left[\frac{e^{-\lambda_\ell t}}{(\lambda_k-\lambda_\ell)(\lambda_j-\lambda_\ell)(\lambda_i-\lambda_\ell)(\lambda_1-\lambda_\ell)} + \frac{e^{-\lambda_k t}}{(\lambda_\ell-\lambda_k)(\lambda_j-\lambda_k)(\lambda_i-\lambda_k)(\lambda_1-\lambda_k)} + \right.$$

$$\frac{e^{-\lambda_j t}}{(\lambda_\ell-\lambda_j)(\lambda_k-\lambda_j)(\lambda_i-\lambda_j)(\lambda_1-\lambda_j)} + \frac{e^{-\lambda_i t}}{(\lambda_\ell-\lambda_i)(\lambda_k-\lambda_i)(\lambda_j-\lambda_i)(\lambda_1-\lambda_i)} +$$

$$\left. \frac{e^{-\lambda_1 t}}{(\lambda_\ell-\lambda_1)(\lambda_k-\lambda_1)(\lambda_j-\lambda_1)(\lambda_i-\lambda_1)} \right] . \quad (3.1d)$$

From any absorption state, the total transition rate is zero, and the time-dependent probability of occupancy can be obtained from Equations (3.1a) through (3.1d) by setting the value of λ for the final step equal to zero. The probability of occupancy of the absorption states for each step of the chain in the limit of infinite times is the following:

$$\text{If } \lambda_i = 0, \lim_{t \to \infty} P_i(t) = \frac{\lambda_{1i}}{\lambda_1}$$

$$\text{If } \lambda_j = 0, \lim_{t \to \infty} P_j(t) = \sum_i \frac{\lambda_{1i}}{\lambda_1} \frac{\lambda_{ij}}{\lambda_i}$$

$$\text{If } \lambda_k = 0, \lim_{t \to \infty} P_k(t) = \sum_i \sum_j \frac{\lambda_{1i}}{\lambda_1} \frac{\lambda_{ij}}{\lambda_i} \frac{\lambda_{jk}}{\lambda_j}$$

$$\text{If } \lambda_\ell = 0, \lim_{t \to \infty} P_\ell(t) = \sum_i \sum_j \sum_k \frac{\lambda_{1i}}{\lambda_1} \frac{\lambda_{ij}}{\lambda_i} \frac{\lambda_{jk}}{\lambda_j} \frac{\lambda_{k\ell}}{\lambda_k} .$$

The object of this analysis is to determine the total probability of occupancy at any time of all states which correspond to a particular outcome of interest. Outcomes of particular concern are: (1) the HVU only is destroyed (states 14, 15, 16, 25, 26, 27, 28, 36, 37 and 38); (2) both the HVU and LVU are destroyed (states 39, 40, 41, 42, 44, 45, 46, 47 and 48); (3) the entire Red force is destroyed or exhausted without destroying the HVU or the LVU (state 32); and (4) the entire Red force is destroyed or exhausted without destroying the HVU, but destroys the LVU (state 43). After the required calculations have been made, an outcome plot like that of Figure 11 can be made.

Transition Rates

A transition within the Markov chain reflects the occurrance of a battle engagement which involves a particular type of Red unit (but not necessarily a specific Blue unit type) and has a particular outcome. However, for an engagement to occur at time t, several conditions must be met. First, in order to have an encounter at all, the Red unit involved must at time t be searching, not stopped to classify a target. Second, a proper encounter, i.e., one which can lead to the specified engagement outcome, must occur within time dt. Third, the proper encounter must result in an engagement with the specified outcome. If we denote the probability that the Red unit involved in the transition is searching at a given time by p_s, the probability that a proper encounter e occurs within time dt by $\lambda_e dt$, and the probability that this

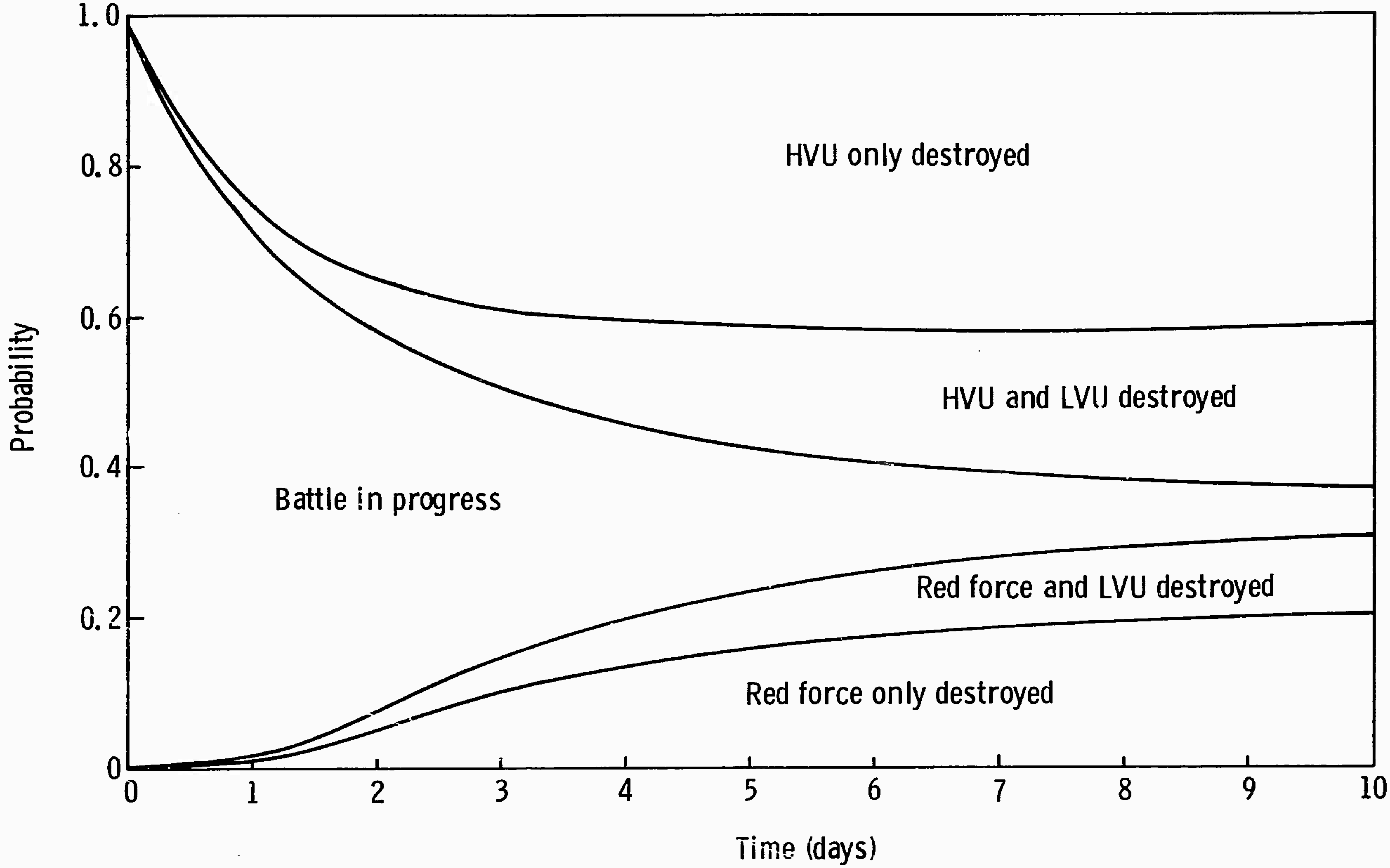

FIGURE 11. OUTCOME PLOTS

encounter leads to an engagement with a particular outcome by p_{eo}, the transition rate λ_{ij} from the initial state i to the final state j corresponding to the specified engagement outcome is

$$\lambda_{ij} = p_s \sum_e \lambda_e p_{eo} \cdot \qquad (3.2)$$

In order to calculate the quantities p_s, λ_e and p_{eo}, and ultimately the transition rates λ_{ij}, the following quantities must be defined and, if numerical results are desired, evaluated:

$\gamma_{i\ell} ds$ = probability that a Red unit of type ℓ will encounter a particular Blue unit of type i within a search element of length ds

$\theta_{a\ell} dt$ = probability that a Red unit of type ℓ will encounter Blue GAF within a time element dt

V_ℓ = speed of Red unit of type ℓ

$h_{i\ell}$ = time required for Red unit of type ℓ to classify a Blue unit of type i

$\delta_{i\ell}$ = probability that a Red unit of type ℓ correctly classifies a Blue unit of type i

$\sigma_{a\ell}$ = probability that a Red unit of type ℓ survives an attack by Blue GAF

$\sigma_{i\ell}$ = probability that a Red unit of type ℓ survives the local defenses of an encountered Blue unit of type i and also, during the identification period $h_{i\ell}$, any attack by Blue GAF. 1/

1/ Destruction of a Red unit by Blue GAF is assumed to be equally probable at all times during the battle. This process can be treated as one which can occur only during search periods, like attrition resulting from encounters with Blue surface units, by including the probability of destruction by Blue GAF during an identification period with the probability of destruction by the Blue unit under inspection. If the probability of surviving the latter defense only is $\sigma_{i\ell}'$, then the effective survival probability of unit ℓ, including the risk it suffers from GAF during period $h_{i\ell}$ is,

$$\sigma_{i\ell} = \sigma_{i\ell}' \left[e^{-\theta_{a\ell} h_{i\ell}} + (1-e^{-\theta_{a\ell} h_{i\ell}})\sigma_{a\ell} \right] = \sigma_{a\ell}' \left[(1-\sigma_{a\ell}) e^{-\theta_{a\ell} h_{i\ell}} + \sigma_{a\ell} \right].$$

$\mu_{i\ell}$ = probability that Blue unit of type i survives attack by a Red unit of type ℓ.

To evaluate $p_{s\ell}$, the probability that a Red unit of type ℓ is searching, one must consider that the operation of a Red unit is divided into periods of search and periods of classification, and that a Red unit begins the battle in the search mode. For any but short periods after the battle begins, during which the latter fact has a complicating effect (c.f., Appendix C), the probability that Red unit ℓ is searching is simply the ratio of the expected interval between encounters with Blue units of any type, $<t_{s\ell}>$, to the sum of this expected interval and the expected classification time $<h_\ell>$, i.e.,

$$p_{s\ell} = \frac{<t_{s\ell}>}{<t_{s\ell}> + <h_\ell>} . \tag{3.3}$$

The expected interval between encounters with Blue units in the present instance is

$$<t_{s\ell}> = \frac{1}{v_\ell \Sigma n_i \gamma_{i\ell}} \tag{3.4}$$

where n_i is the number of operational Blue units of type i. For the present example n_0 is always 1 and n_1 is either 1 or 0. The expected classification period is the sum of the required classification periods for all Blue unit types, each weighted according to the relative frequencies of encounter, i.e.,

$$<h_\ell> = \frac{\sum_i h_{i\ell} n_{i\ell} \gamma_{i\ell}}{\sum_i n_i \gamma_{i\ell}} . \tag{3.5}$$

Thus,

$$p_{s\ell} = \frac{1/v_\ell}{1/v_\ell + \Sigma h_{i\ell} n_i \gamma_{i\ell}} . \tag{3.6}$$

To evaluate λ_e, the time rate of occurrence of a specific proper encounter e, given that the Red unit involved is searching, one simply multiplies the rate per unit distance searched at which the proper encounter e occurs by the speed of the Red unit v_ℓ. The value of p_{eo} is the total probabilities of all events or combinations of events which can occur for the encounter to become an engagement with the required outcome.

Determination of the entire set of 94 transition rates which are pertinent to this simplified battle requires the evaluation of only 15 independent

transition rates, the remaining transition rates being expressible in terms of these. The independent transition rates from states in which the LVU has not been destroyed, expressed in terms of the quantities defined earlier, are

$$\lambda_{2,7} = \frac{\frac{\theta_{a_3}}{v_3}(1-\sigma_{a_3}) + \gamma_{03}[(1-\sigma_{03}) + \sigma_{03}\,\delta_{03}\,\mu_{03}] + \gamma_{13}[(1-\sigma_{13}) + \sigma_{13}(1-\delta_{13})\mu_{13}]}{1/v_3 + h_{03}\gamma_{03} + h_{13}\gamma_{13}}$$

$$\lambda_{1,3} = \frac{\frac{\theta_{a_1}}{v_1}(1-\sigma_{a_1}) + \gamma_{01}[(1-\sigma_{01}) + \sigma_{01}\,\delta_{01}\,\mu_{01}] + \gamma_{11}[(1-\sigma_{11}) + \sigma_{11}(1-\delta_{11})\mu_{11}]}{1/v_1 + h_{01}\gamma_{01} + h_{11}\gamma_{11}}$$

$$\lambda_{1,4} = \frac{\frac{\theta_{a_2}}{v_2}(1-\sigma_{a_2}) + \gamma_{02}[(1-\sigma_{02}) + \sigma_{02}\,\delta_{02}\,\mu_{02}] + \gamma_{12}[(1-\sigma_{12}) + \sigma_{12}(1-\delta_{12})\mu_{12}]}{1/v_2 + h_{02}\gamma_{02} + h_{12}\gamma_{12}}$$

$$\lambda_{2,21} = \frac{\gamma_{13}\,\sigma_{13}(1-\delta_{13})(1-\mu_{13})}{1/v_3 + h_{03}\gamma_{03} + h_{13}\gamma_{13}} \qquad \lambda_{1,12} = \frac{\gamma_{11}\,\sigma_{11}(1-\delta_{11})(1-\mu_{11})}{1/v_1 + h_{01}\gamma_{01} + h_{11}\gamma_{11}}$$

$$\lambda_{1,13} = \frac{\gamma_{12}\,\sigma_{12}(1-\delta_{12})(1-\mu_{12})}{1/v_2 + h_{02}\gamma_{02} + h_{12}\gamma_{12}} \qquad \lambda_{2,25} = \frac{\gamma_{03}\,\sigma_{03}\,\delta_{03}(1-\mu_{03})}{1/v_3 + h_{03}\gamma_{03} + h_{13}\gamma_{13}}$$

$$\lambda_{1,15} = \frac{\gamma_{01}\,\sigma_{01}\,\delta_{01}(1-\mu_{01})}{1/v_1 + h_{01}\gamma_{01} + h_{11}\gamma_{11}} \qquad \lambda_{1,15} = \frac{\gamma_{02}\,\sigma_{02}\,\delta_{02}(1-\mu_{02})}{1/v_2 + h_{02}\gamma_{02} + h_{12}\gamma_{12}}\,.$$

The independent transition rates from states in which the LVU has been destroyed are:

$$\lambda_{11,21} = \frac{\frac{\theta_{a_3}}{v_3}(1-\sigma_{a_3}) + \gamma_{03}(1-\sigma_{03} + \sigma_{03}\,\delta_{03}\,\mu_{03})}{1/v_3 + h_{03}\gamma_{03}}$$

$$\lambda_{11,22} = \frac{\frac{\theta_{a_1}}{v_1}(1-\sigma_{a_1}) + \gamma_{01}(1-\sigma_{01}+\sigma_{01}\delta_{01}\mu_{01})}{1/v_1 + h_{01}\gamma_{01}}$$

$$\lambda_{11,23} = \frac{\frac{\theta_{a_2}}{v_2}(1-\sigma_{a_2}) + \gamma_{02}(1-\sigma_{02}+\sigma_{02}\delta_{02}\mu_{02})}{1/v_2 + h_{02}\gamma_{02}}$$

$$\lambda_{11,39} = \frac{\gamma_{03}\sigma_{03}\delta_{03}(1-\mu_{03})}{1/v_3 + h_{03}\gamma_{03}} \qquad \lambda_{11,40} = \frac{\gamma_{01}\sigma_{01}\delta_{01}(1-\mu_{01})}{1/v_1 + h_{01}\gamma_{01}}$$

$$\lambda_{11,41} = \frac{\gamma_{02}\sigma_{02}\delta_{02}(1-\mu_{02})}{1/v_2 + h_{02}\gamma_{02}} .$$

The remaining transition rates are

$$\lambda_{8,18} = \lambda_{9,19} = \lambda_{20,32} = \lambda_{2,7}$$

$$\lambda_{1,2} = \lambda_{3,8} = \lambda_{4,9} = \lambda_{10,20} = 2\lambda_{2,7}$$

$$\lambda_{2,8} = \lambda_{4,10} = \lambda_{7,18} = \lambda_{9,20} = \lambda_{19,32} = \lambda_{1,3}$$

$$\lambda_{2,9} = \lambda_{3,10} = \lambda_{7,19} = \lambda_{8,20} = \lambda_{18,32} = \lambda_{1,4}$$

$$\lambda_{8,33} = \lambda_{9,34} = \lambda_{20,43} = \lambda_{2,21}$$

$$\lambda_{1,11} = \lambda_{3,22} = \lambda_{4,23} = \lambda_{10,35} = 2\lambda_{2,21}$$

$$\lambda_{2,22} = \lambda_{7,33} = \lambda_{4,24} = \lambda_{9,35} = \lambda_{19,43} = \lambda_{1,12}$$

$$\lambda_{2,23} = \lambda_{7,34} = \lambda_{3,24} = \lambda_{8,35} = \lambda_{18,43} = \lambda_{1,13}$$

$$\lambda_{8,36} = \lambda_{9,37} = \lambda_{20,44} = \lambda_{2,25}$$

$$\lambda_{1,14} = \lambda_{4,27} = \lambda_{10,38} = \lambda_{3,28} = 2\lambda_{2,25}$$

$$\lambda_{2,26} = \lambda_{7,36} = \lambda_{4,29} = \lambda_{9,38} = \lambda_{19,44} = \lambda_{1,15}$$

$$\lambda_{2,27} = \lambda_{3,28} = \lambda_{7,37} = \lambda_{8,38} = \lambda_{18,44} = \lambda_{1,16}$$

$$\lambda_{22,33} = \lambda_{23,34} = \lambda_{35,43} = \lambda_{11,21}$$

$$\lambda_{13,23} = \lambda_{24,35} = \lambda_{12,22} = 2\lambda_{11,21}$$

$$\lambda_{21,33} = \lambda_{13,24} = \lambda_{23,35} = \lambda_{34,43} = \lambda_{11,22}$$

$$\lambda_{12,24} = \lambda_{21,34} = \lambda_{22,35} = \lambda_{33,43} = \lambda_{11,23}$$

$$\lambda_{35,48} = \lambda_{22,45} = \lambda_{23,46} = \lambda_{11,39}$$

$$\lambda_{12,40} = \lambda_{13,41} = \lambda_{24,47} = 2\lambda_{11,39}$$

$$\lambda_{13,42} = \lambda_{21,45} = \lambda_{23,47} = \lambda_{34,48} = \lambda_{11,40}$$

$$\lambda_{12,42} = \lambda_{21,46} = \lambda_{22,47} = \lambda_{33,48} = \lambda_{11,41} .$$

Calculation of the probabilities that the system at any time will be in absorption states corresponding to each of the four battle outcomes mentioned earlier have been made assuming the values given in Table 4 for the encounter rates, survival and correct classification probabilities, identification periods and Red unit speeds. The results are shown in Figure 11.

TABLE 4

ASSUMED BATTLE PARAMETERS

	Red Unit Type, ℓ 1	2	3
$\gamma_{o\ell}$	0.001 nm^{-1}	0.001 nm^{-1}	0.002 nm^{-1}
$\gamma_{1\ell}$	0.002 nm^{-1}	0.002 nm^{-1}	0.002 nm^{-1}
$\theta_{a\ell}$	0.50 day^{-1}	0.25 day^{-1}	0.10 day^{-1}
$\sigma_{o\ell}$	0.5	0.2	0.8
$\sigma_{1\ell}$	0.5	0.2	0.9
$\sigma_{a\ell}$	0.0	0.0	0.5
$\mu_{o\ell}$	0.4	0.2	0.2
$\mu_{1\ell}$	0.6	0.3	0.3
$\delta_{o\ell}$	0.7	0.6	0.7
$\delta_{1\ell}$	0.7	0.6	0.7
$h_{o\ell}$	0.0 day	0.0 day	0.25 day
$h_{1\ell}$	0.0 day	0.0 day	0.25 day
v_{ℓ}	400 nm/day	400 nm/day	200 nm/day

IV. COORDINATED AIR-SEA ATTACK

THE NATURE OF THE BATTLE

In this battle, it is assumed that available Red aircraft could destroy all HVUs of the Blue force if prior reconnaissance efforts had succeeded so that the raid aircraft would not dissipate much of their effectiveness in attacks on LVUs. However, successful reconnaissance is considered sufficiently difficult that it is felt to be desirable for Red sea forces to attempt to penetrate the Blue air barrier and to seek out Blue HVUs and hold contact on them until the prearranged time of the air attack. If the various Red sea units are assigned a value representing the number of raid aircraft to which they are considered equivalent, the sea units holding contact on an HVU can be combined with the number of surviving Red aircraft attacking the HVU to determine the extent of the threat to which the HVU is subjected. Moreover, if the survivability of the HVU is described by a "hardness number" which is the number of raid aircraft equivalents whose simultaneous attack it can withstand, the question of whether the HVU survives or not can be decided on the basis of the success achieved by Red sea units in making and maintaining contacts with the HVU and the results of efforts by Red raid aircraft to locate the HVU without prior reconnaissance. A major contribution of the Red sea units is to enable the Red air force to subdivide itself so as to increase the chance of encountering the HVUs and, if the raid aircraft are supplemented by enough sea units holding contact, destroying them.

HOLDING CONTACT

In this section, the probability of a given Red sea unit holding contact with a Blue HVU at any time after the start of search will be determined. It will be assumed that prior to the establishment of contact (encounter and classification) with what the Red unit ultimately decides

is an HVU, the Red unit is subject to all of the sources of attrition described in Section III, i.e., the local defenses of the Blue unit encountered and the Blue GAF. Once a contact is established, however, only attrition from the GAF is operative. Since this Blue defense would, over a long enough time, destroy any Red unit, the probability that the Red unit holds contact with an HVU increases with time from its initial value of zero, passes through a maximum, and ultimately decreases again to zero. This maximum probability of holding contact and the time at which it occurs are quantities of particular interest and will be calculated.

At all but short times after the start of search, the occurrence of encounters which, after a classification attempt, result in the establishment of a contact, or else in the destruction of the Red unit, are Poisson distributed. Let the parameters of the Poisson distribution for various engagement results be the following:

G = destruction of the Red unit by GAF

D = destruction of the Red unit by either an HVU or an LVU

H = establishment of contact with an HVU, correctly classified

L = establishment of contact with an LVU misclassified as an HVU.

These parameters can be expressed in terms of the quantities given in Section III. The appropriate expressions for any Red unit of type ℓ and Blue unit of type 0 (HVU) or 1 (LVU) are:

$$G_\ell = \theta_{a\ell}(1-\sigma_{a\ell})$$

$$D_\ell = \frac{n_0\gamma_{0\ell}\{1-\sigma_{0\ell}/[(1-\sigma_{a\ell})e^{-\theta_{a\ell}h_{0\ell}}+\sigma_{a\ell}]\}+n_1\gamma_{1\ell}\{1-\sigma_{1\ell}/[(1-\sigma_{a\ell})e^{-\theta_{a\ell}h_{1\ell}}+\sigma_{a\ell}]\}}{1/v_\ell+h_{0\ell}n_0\gamma_{0\ell}+h_{1\ell}n_1\gamma_{1\ell}}$$

$$H_\ell = \frac{n_0\gamma_{0\ell}\,\delta_{0\ell}\,\sigma_{0\ell}/[(1-\sigma_{a\ell})e^{-\theta_{a\ell}h_{0\ell}}+\sigma_{a\ell}]}{1/v_\ell+h_{0\ell}n_0\gamma_{0\ell}+h_{1\ell}n_1\gamma_{1\ell}}$$

$$L_\ell = \frac{n_1\gamma_{1\ell}(1-\delta_{1\ell})\sigma_{1\ell}/[(1-\sigma_{a\ell})e^{-\theta_{a\ell}h_{1\ell}}+\sigma_{a\ell}]}{1/v_\ell+h_{0\ell}n_0\gamma_{0\ell}+h_{1\ell}n_1\gamma_{1\ell}}.$$

The probability that at time t the Red unit has not been destroyed by GAF is e^{-GT}. The probability that it has not established a contact with either an HVU or an LVU, or has not been destroyed by these Blue units is $e^{-(H+L+D)t}$, and thus the probability that it has experienced one or another of these mutually exclusive events is $1 - e^{-(H+L+D)t}$. Specifically the probability that the Red unit has established contact with an HVU and is still maintaining it at time t is

$$P_H(t) = \frac{H}{H+L+D}\left(1 - e^{-(H+L+D)t}\right)e^{-Gt}. \qquad (4.1)$$

Similarly, the probability that the Red unit is holding contact on an LVU is

$$P_L(t) = \frac{L}{H+L+D}\left(1 - e^{-(H+L+D)t}\right)e^{-Gt}. \qquad (4.2)$$

The probability that the Red unit has been destroyed is one minus the probability that it has not been destroyed by either the Blue surface units or by the GAF, i.e.,

$$P_D(t) = 1 - \left[1 - \frac{D}{H+L+D}\left(1 - e^{-(H+L+D)t}\right)\right] e^{-Gt}. \qquad (4.3)$$

The remaining possibility, the null event representing survival but no contacts held for the Red unit, has a probability

$$P_o(t) = e^{-(H+L+D+G)t}. \qquad (4.4)$$

The maximum value of P_H and the time at which this value is achieved are obtained by setting the time derivative of P_H equal to zero, solving for the time t_o at which this occurs, and evaluating $P_H(t_o)$. Thus,

$$\left.\frac{dP_H}{dt}\right|_{t_o} = \frac{H}{H+L+D}\left[-Ge^{-Gt_o} + (G+H+L+D)e^{-(G+H+L+D)t_o}\right] = 0$$

for

$$t_o = \frac{1}{H + L + D} \ln \frac{G + H + L + D}{G},$$

and

$$P_H(t_o) = \frac{H}{G + H + L + D} \left(\frac{G}{G + H + L + D} \right)^{\frac{G}{H + L + D}} . \tag{4.5}$$

The fraction of its maximum value that $P_H(t)$ assumes over a range of times is shown in Figure 12 for several values of G/Z, where $Z = G + H + L + D$. Time is measured in units of $1/Z$.

AN EXAMPLE

Suppose the Red force described in Section III attempts to aid a Red air force in attacking the Blue force of Section III by holding contact on the HVU until the time of the air attack. It will be assumed that each Red unit begins its search at such a time that its probability of holding contact on the HVU at the time of the scheduled air attack is a maximum.

The first step is to determine the parameters G_ℓ, H_ℓ, L_ℓ, D_ℓ in terms of the quantities assumed in Section III. These parameter values are given in Table 5. Also given in this table are the maximum probabilities that a Red unit of type ℓ holds contact on the HVU, $P_{H\ell}(t_o)$, and the numbers of aircraft to which each Red unit type is assumed to be equivalent, AE_ℓ.

TABLE 5

CALCULATED PARAMETERS AND ASSUMED AIRCRAFT EQUIVALENTS

	G_ℓ (days^{-1})	H_ℓ (days^{-1})	L_ℓ (days^{-1})	D_ℓ (days^{-1})	$P_{H\ell}(t_o)$	AE_ℓ
1	0.5000	0.1400	0.1200	0.6000	0.0575	6
2	0.2500	0.0480	0.0640	0.6000	0.0319	3
3	0.0500	0.1890	0.0911	0.0929	0.3356	3

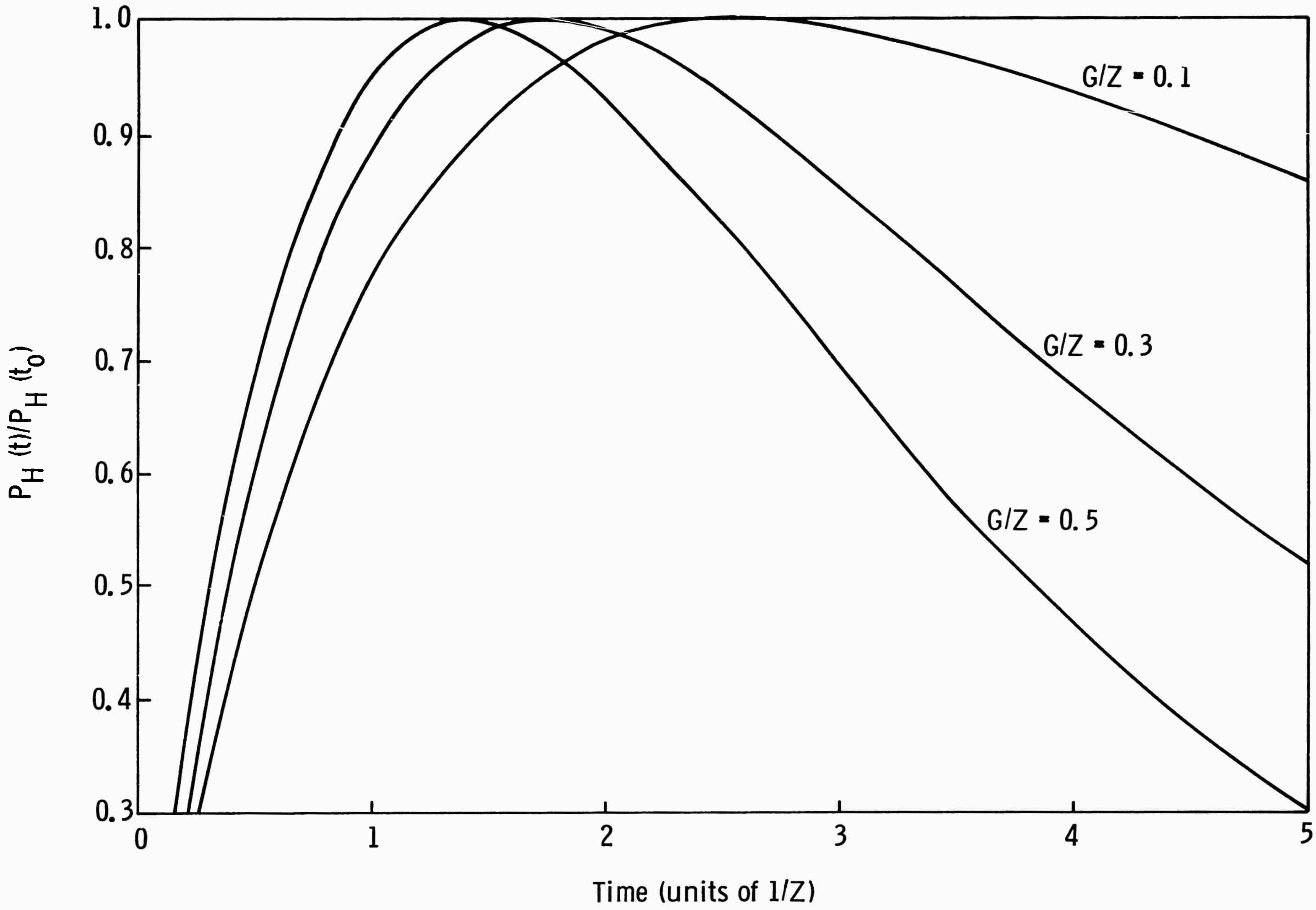

FIGURE 12. FRACTION OF MAXIMUM PROBABILITY OF RED UNIT HOLDING HVU CONTACT

The second step is to determine the probability of various units holding contact on the HVU at the time of attack. These probabilities for various combinations of units, listed in order of decreasing number of their aircraft equivalents, together with the probability of having at least a certain number n of aircraft equivalents AE holding contact on the HVU, are given in Table 6.

TABLE 6

PROBABILITIES OF RED UNIT COMBINATIONS HOLDING HVU AND OF PRESENCE OF VARIOUS AIRCRAFT EQUIVALENTS

Combination	$P_{\text{Combination}}$	$\sum_{\ell} AE_{\ell}$	n	$P(AE \geq n)$
1, 2, 3^a	0.0002	15	15	0.0002
1, 2, 3	0.0012	12		
1, 3^a	0.0065	12	12	0.0079
2, 3^a	0.0036	9		
1, 2	0.0018	9		
1, 3	0.0386	9	9	0.0519
2, 3	0.0214	6		
3^a	0.1126	6		
1	0.0575	6	6	0.2435
2	0.0319	3		
3	0.6712	3	3	0.9466

The final step is to determine the probability that the HVU survives attack for various Red possibilities of dividing its air force into two searching units, only one of which being assumed to find and attack the HVU. Destruction of the HVU will be decided by the participation of y or more Red aircraft equivalents in the attack. It will be assumed that the total Red air force consists of 13 aircraft, and that the section which attacks the HVU will itself be attacked by 10 Blue interceptors. These interceptors will reduce the attacking Red section of size x to a size x' according to an assumed attrition law

$$x' = xe^{-\alpha y/x}, \tag{4.6}$$

where y is the number of Blue interceptors and α is a constant. For the purposes of this example, a value $\alpha = \ln 1.5$ will be assumed. Each possibility of subdividing the Red air force will generate two equally likely numbers of aircraft, x_1' and x_2', which survive Blue air defenses and attack the HVU. The probabilities that Red sea units holding contact on the HVU can supplement x_1' oe x_2' sufficiently to cause destruction of the HVU are s_1 and s_2, respectively. These numbers can be found in Table 6, since $s_1 = P_{AE \geq 7 - x_1'}$ and $s_2 = P_{AE \geq 7 - x_2'}$. The probability that the HVU is destroyed, P_k, is

$$P_k = .5\, s_1 + .5\, s_2 . \tag{4.7}$$

Values of x_1', x_2', s_1', s_2 and P_k for each strategy of division of the Red air force are given in Table 7.

TABLE 7

PROBABILITY OF DESTROYING HVU FOR VARIOUS RED AIR ATTACK STRATEGIES

Strategy	x_1'	x_2'	s_1	s_2	P_k
(13, 0)	9.517	0.000	1.0000	0.0519	0.5260
(12, 1)	8.559	0.017	1.0000	0.0519	0.5260
(11, 2)	7.609	0.263	1.0000	0.0519	0.5260
(10, 3)	6.666	0.777	0.9466	0.0519	0.4993
(9, 4)	5.736	1.452	0.0466	0.2435	0.5951
(8, 5)	4.819	2.222	0.0466	0.2435	0.5951
(7, 6)	3.922	3.052	0.2435	0.2435	0.2435

It is clear from these results that, in the absence of successful reconnaissance, the optimum strategies for the Red air force are the divisions (9,4) and (8,5). The value of P_k for these strategies can be instructively contrasted with that for the division (11,2). The latter strategy might be selected if the Red air force, while acknowledging the possibility of assistance by sea units, neglected to take specific account of the probability of such assistance in the planning of its attack.

V. CONCLUSION AND RECOMMENDATIONS

The major conclusion of this study effort is that a methodology has been developed which will enable Navy systems and tactical planners to evaluate, in a multi-threat environment, the effectiveness of current naval Carrier Task Forces to successfully counter air, surface, and sub-surface attacks whether independently launched or coordinated. Furthermore, the methodology developed in this research effort can be utilized to assess the potential contribution to carrier defense of new and/or proposed systems.

It is recommended that:

a. A research effort be undertaken to find the ranges of the parameter values of the methodology from existing operation data and from these data an assessment be made of current Carrier Task Force vulnerability

b. The methodology be applied to evaluate potential contributions to Carrier Task Force effectiveness of new and/or proposed defense systems under various tactical assumptions

c. The methodology be exercised to find the sensitivity of the outcome to various parameter values and to search for those important parameters which drive the answer

d. Simplified computational procedures be developed for dealing with generalized battle situations.

APPENDIX A

SURVIVOR LIMITS IN THE AIR/SURFACE ENGAGEMENT

Statement of the problem: Suppose that the variables x_0, x_1, $x_2, \ldots$ and y_0, y_1, $y_2, \ldots$ are related by

$$x_{n+1} = x_n \exp(-\alpha y_n/x_n) \tag{A.1}$$

$$y_{n+1} = y_n \exp(-\beta x_n/y_n) \tag{A.2}$$

for $n = 0, 1, 2, \ldots$, where α, β, x_0, y_0 are given positive constants. It follows that x_1, $x_2, \ldots$ and y_1, $y_2, \ldots$ are also positive. What are the limits of x_n, y_n as $n \to \infty$?

Define

$$R_n = y_n/x_n = 1/S_n \tag{A.3}$$

$$r_0 = R_0 = 1/s_0 \tag{A.4}$$

$$r_{n+1} = \exp(-\beta/R_n)/\exp(-\alpha R_n) = 1/s_{n+1}. \tag{A.5}$$

Lemma

$$R_n = r_0 r_1 \ldots r_n \tag{A.6}$$

$$x_n/x_0 = \exp(-\alpha \Sigma_0^{n-1} R_i). \tag{A.7}$$

Proof: Divide the respective members of equation (A.2) by those of (A.1):

$$R_{n+1} = r_{n+1} R_n \tag{A.8}$$

and equation (A.6) follows by induction. To derive equation (A.7) note that

$$x_n/x_0 = \Pi_0^{n-1} x_{i+1}/x_i,$$

since the product telescopes. But

$$x_{i+1}/x_i = \exp(-\alpha y_i/x_i) = \exp(-\alpha R_i),$$

first by equation (A.1) second by equation (A.3).

Thus,

$$x_n/x_0 = \Pi_0^{n-1} \exp(-\alpha R_i) = \exp(-\alpha \Sigma_0^{n-1} R_i), \quad \text{q.e.d.}$$

Similarly,

$$S_n = s_0 s_1 \ldots s_n \tag{A.9}$$

and

$$y_n/y_0 = \exp(-\beta \Sigma_0^{n-1} S_i). \tag{A.10}$$

Theorem:

If $r_0^2 < \beta/\alpha$, then

$1 > r_1 > r_2 > \ldots$

$1 < s_2 < \ldots$

$\lim R_n = 0$, $\lim S_n = \infty$, $\lim r_n = 0$, $\lim s_n = \infty$

$0 < \lim x_n < \infty$

$\lim y_n = 0.$

If $r_o^2 = \beta/\alpha$, then

$$1 = \lambda_1 = r_2 = \ldots$$

$$1 = s_1 = s_2 = \ldots$$

$$\lim R_n^2 = \beta/\alpha, \ \lim S_n^2 = \alpha/\beta, \ \lim r_i = 1, \ \lim s_n = 1$$

$$\lim x_n = 0$$

$$\lim y_n = 0.$$

If $r_o^2 > \beta/\alpha$, then

$$1 < r_1 < r_2 < \ldots$$

$$1 > s_1 > s_2 > \ldots$$

$$\lim R_n = \infty, \ \lim S_n = 0, \ \lim r_n = \infty, \ \lim s_n = 0$$

$$\lim x_n = 0$$

$$0 < \lim y_n < \infty.$$

Proof: Let $r_o^2 < \beta/\alpha$, then $r_1 < 1$ by equation (A.5), the definition of r_1. This in turn implies that $r_2 < r_1$, so that $1 > r_1 > r_2 > \ldots$ is established by induction. The assertor $1 < s_1 < s_2 < \ldots$ is similarly established, and the rest of the assertions for this case ($r_o^2 < \beta/\alpha$) are immediate. The other two parts of the theorem are established in the same manner q.e.d.

APPENDIX B

OPTIMAL RED MISSILE CRUISER ALLOCATION

Statement of the problem: Suppose RED initially allocates X missile cruisers and suppose that the probability, V, of RED mission failure is

$$V = P^{X'} \tag{B.1}$$

where P is a probability and X' is the number of missile cruisers which are undetected by the BLUE outer defense barrier. The strategies available to RED are the partitions of X; that is, sets $j_1, j_2, \ldots$ and $n_1, n_2, \ldots$ of natural numbers such that

$$n_1 j_1 + n_2 j_2 \tau \ldots = X \text{ and } j_1 > j_2 > \ldots \tag{B.2}$$

gives a partition of this form, the probability, p_i, that X' = i is

$$p_i = \Sigma (1-E)^m E^{n-m} \Pi_k \binom{n_k}{m_k} \tag{B.3}$$

where $0 < E < 1$ characterizing the barrier detection capability, $m = \Sigma_k m_k$, $n = \Sigma_k n_k$, and the summation is taken over all $(m_1, m_2 \ldots)$ satisfying $m_k = 0, 1, \ldots, n_k$ and $\Sigma m_k j_h = i$. Which partition should RED use in order to minimize the expected value of V, E(V)?

From Equations B.1 and B.3 it follows that

$$E(V) = \Pi_k \left[e + (1-e) P^{j_k}\right]^{n_k} \tag{B.4}$$

consider

$$e + (1-e) P < (e + (1-e) P^j)^{1/j} \tag{B.5}$$

for j = 2, 3, ...

This states that the mean value of 1 and P (with weight e and 1-e, respectively) is less than the jth root of the weighted sum of the jth powers of 1 and P for $j > 1$. This is well known. From Equation B.5 it follows that

$$(e + (1-e)\,P)^j < e + (1-e)\,P^j \tag{B.6}$$

Theorm: E(V) is minimized by the partition 1^X; that is a collection of 1's.

Proof: Since by Equation B.4.

$$E(V) = \Pi_k \left[e + (1-e)\,P^{j_k}\right]^{n_k} \tag{B.7}$$

any $j_k > 1$ could be replaced, by Equation B.6, making $j_k = 1$ and replacing n_k by $n_k j_k$. The theorm is thus established by induction.

Generalization: If e is not fixed but is a nondecreasing function of raid group size, the theorem is still true. That is, if the optimal size of raid groups is 1 for equal detectability, then a fortiori it is optimal if detectability increases with raid group size.

APPENDIX C

PROBABILITY OF SEARCH MODE AT SHORT TIMES

Suppose that an expected interval between encounters, $\langle t_s \rangle$, and an expected classification period $\langle h \rangle$, both as defined in Section III, characterize the operation of a Red unit, and that during search periods encounters of any type are Poisson distributed with parameter $1/\langle t_s \rangle$. If the Red unit is searching at time t and has had n prior encounters, these n encounters must have occurred during a total search time $t - n\langle h_s \rangle$. Since the encounters are Poisson distributed, the probability that exactly n encounters have occurred within time $t - n\langle h \rangle$ is

$$P_n(t - n\langle h \rangle) = \left[\frac{t - n\langle h \rangle}{\langle t_s \rangle}\right]^n \frac{e^{-\frac{t - n\langle h \rangle}{\langle t_s \rangle}}}{n!} . \tag{C.1}$$

The probability that the unit is searching at time t is therefore the sum

$$P_s(t) = \sum_{n=0}^{m} \left[\frac{t - n\langle h \rangle}{\langle t_s \rangle}\right]^n \frac{e^{-\frac{t - n\langle h \rangle}{\langle t_s \rangle}}}{n!} \tag{C.2}$$

where the limit m is the largest integer for which $t - m\langle h \rangle$ is positive.

The function $P_s(t)$ is plotted in Figure C.1 for several values of $<t_s>$. Time is expressed in units of $<h>$. It can be seen that, as intuition would lead one to expect,

$$\lim_{t \to \infty} P_s(t) = \frac{<t_s>}{<t_s> + <h>} \; . \qquad \text{(C.3)}$$

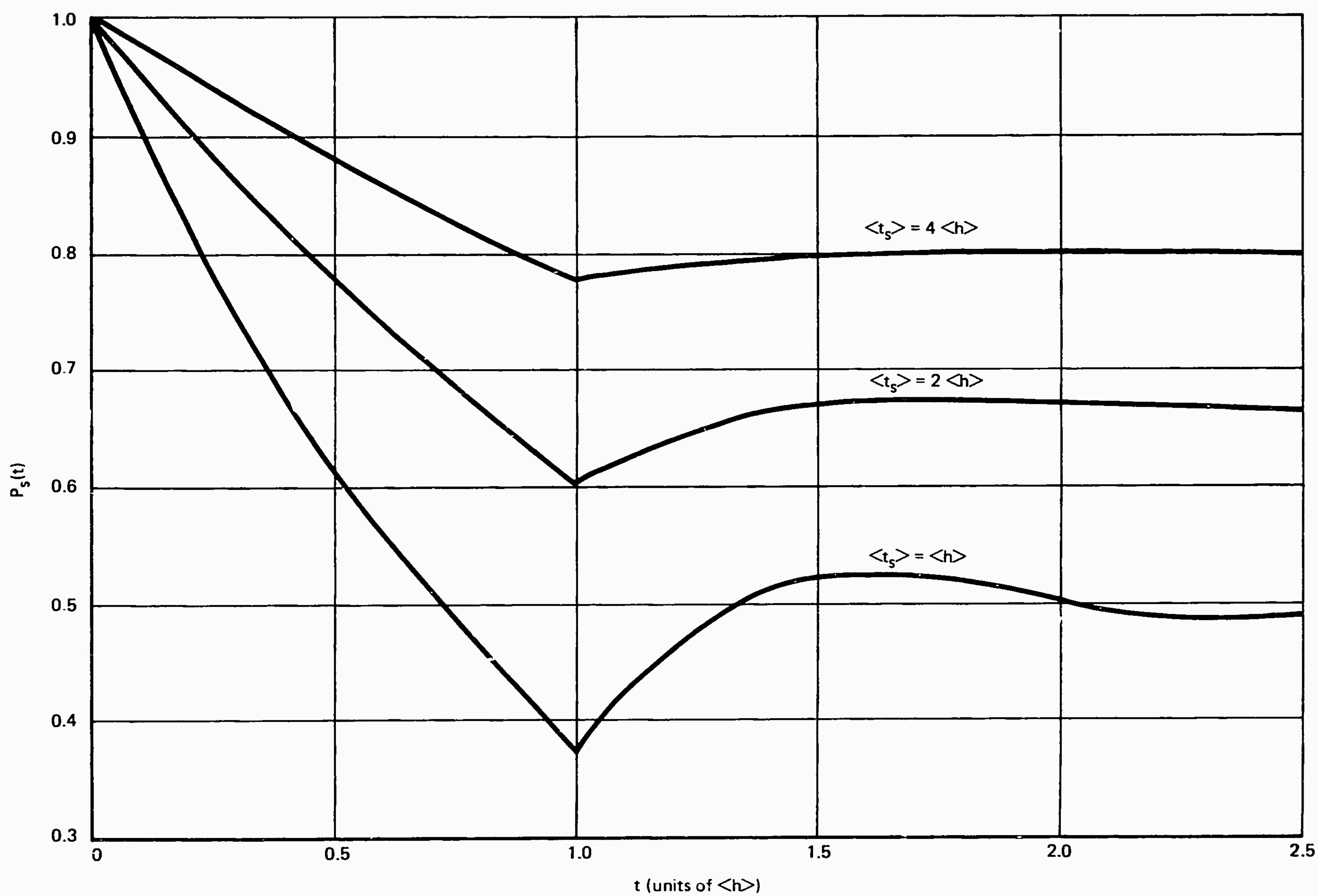

FIGURE C.1. PROBABILITY OF RED UNIT IN SEARCH MODE